Spectral problems for matrix ODOs (MODOs) and Picard-Vessiot Theory

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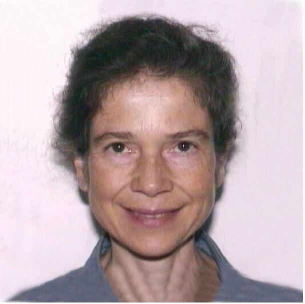
I will present recent work
with M.A. Zurro and E. Previato.

In Memory of Emma Previato (1952-2022)



It was a vision of E. Previato the convenience of a triple approach combining differential algebra, Picard-Vessiot extensions and representation theory to study spectral problems for commuting differential operators.

Motivation

DIFFERENTIAL OPERATORS $\Longleftrightarrow$ ALGEBRAIC CURVES

- ODOs (scalar coefficients): Burchnall-Chaundy, Baker, Krichever...
- MODOs (matrix coefficients): Krichever, Wilson, Grinevich, Mulase...

Direct problem $\Longrightarrow$

Inverse problem $\Longleftarrow$

Contents

MODOs

(K, ∂), constants $C = \overline{C}$ of char 0: $\mathbb{C}(x)$, $\mathbb{C}(e^x)$, $\partial = d/dx$

$$\mathcal{R}_\ell = M_\ell(K) \text{ ring of } \ell \times \ell \text{ matrices}$$

Derivation D in $\mathcal{R}_\ell$. For $A = (a_{\alpha,\beta}) \in \mathcal{R}_\ell$,

$$D(A) := A' = (a'_{\alpha,\beta})$$

Matrix Ordinary Differential Operators or MODOs

$$\mathcal{R}_\ell[D]$$

Non commutative ring $DA := AD + A'$

$$K[\partial] \hookrightarrow \mathcal{R}_\ell[D] \text{ by } \sum a_i \partial^i \mapsto \sum a_i I_\ell D^i$$

AKNS (1974, Ablowitz, Kaup, Newell and Segur)

Previato, E. (1985). Hyperelliptic quasi-periodic and soliton solutions of the nonlinear Schrödinger equation. Duke Math.

$K = \mathbb{C}\langle u, v\rangle$, $\mathcal{R}_2[D] = M_2(K)[D]$

Stationary AKNS: $\frac{\imath}{2} v_{xx} + \imath v^2 u = 0$, $\frac{\imath}{2} u_{xx} + \imath v u^2 = 0$

$$L = \imath \begin{bmatrix} D & u \\ v & -D \end{bmatrix} = A_0 + A_1 D, \text{with } A_0 = \imath \begin{bmatrix} 0 & u \\ v & 0 \end{bmatrix}, A_1 = \imath \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

$$B = \imath \begin{bmatrix} -2D^2 - uv & -2uD - u_x \\ -2vD - v_x & 2D^2 + uv \end{bmatrix} = B_0 + B_1 D + B_2 D^2 ,$$

where

$$B_0 = \imath \begin{bmatrix} -uv & -u_x \\ -v_x & uv \end{bmatrix} , \quad B_1 = \imath \begin{bmatrix} 0 & -2u \\ -2v & 0 \end{bmatrix} , \quad B_2 = \imath \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix} .$$

Spectral Problem

Given L and B in $\mathcal{R}_\ell[D]$

$$LY = \lambda Y \quad , \quad BY = \mu Y \quad , \quad Y = (y_1, \ldots, y_\ell)^t.$$

for

$$L = A_0 + A_1 D \quad \text{and} \quad B = \sum_{j=0}^{n} B_j D^j, \text{ where } n \geq 1 \ ,$$

assuming that A_1 is invertible.

$$\partial\lambda = 0, \ \partial\mu = 0$$

$$P = L - \lambda := L - \lambda I_\ell \text{ and } Q = B - \mu := B - \mu I_\ell$$

- Wilson, G. (1979). Commuting flows and conservation laws for Lax equations.
- Krichever, I. M. (1976). Algebraic curves and commuting matricial differential operators.
- Grinevich, P. G. (1987). Vector Rank of Commuting Matrix differential operators. Proof of S. P. Novikov's criterion.
- Oganesyan, V. (2019). Matrix Commuting Differential Operators of Rank 2 and Arbitrary Genus.

Contents

Common solutions

We look for a necessary and sufficient condition on coefficient matrices for

$$\begin{cases} PY = \overline{0} \\ QY = \overline{0} \end{cases}, \quad Y = (y_1, \ldots, y_\ell)^t, \; \overline{0} = (0, \ldots, 0)^t, \qquad (1)$$

to have a common nontrivial solution $\psi = (\psi_1, \ldots, \psi_\ell)^t$, with all the ψ_i in some differential extension Σ of K.

For $P = A_0 + A_1 D$, rewrite systen $PY = \overline{0}$ as

$$DY = NY \quad \text{with} \quad N = -A_1^{-1} A_0 \in \mathcal{R}_\ell. \qquad (2)$$

Σ Picard-Vessiot extension of K for (2).

Given a solution $\psi = (\psi_1, \ldots, \psi_\ell)^t \in \Sigma^\ell$ of system (2), the derivation is defined by $D\psi = N\psi$.

$$D^j\psi = p_j(N)\psi \quad , \quad j \geq 1,$$

with $p_j(N)$ defined by

$$p_0(N) := I_\ell \ , \quad p_j(N) := p_{j-1}(N)N + (p_{j-1}(N))' \ , \quad j \geq 1, \tag{3}$$

For $Q = \sum_{j=0}^n B_j D^j$, it holds,

$$Q\psi = M(P, Q)\psi. \tag{4}$$

with $M(P, Q)$ the $\ell \times \ell$ matrix in $\mathcal{R}_\ell$ defined by

$$M(P, Q) := \sum_{j=0}^{n} B_j p_j(N). \tag{5}$$

Matrix differetial resultant

P and Q in $\mathcal{R}_\ell[D]$, with

$$\mathcal{R}_\ell = M_\ell(K), \quad K^\partial = C = \overline{C}$$

The `matrix differential resultant` of P and Q, for $P = A_0 + A_1 D$, $|A_1| \neq 0$

$$\mathrm{DRes}(P, Q) := \det M(P, Q). \tag{6}$$

Theorem A. *The following statements hold:*

1. *If there exists a common nontrivial solution of* $PY = \overline{0}$ *and* $QY = \overline{0}$ *then* $\mathrm{DRes}(P, Q) = 0$.
2. *If* P *and* Q *commute, and* $\mathrm{DRes}(P, Q) = 0$, *then the matrix differential system* $PY = \overline{0}$, $QY = \overline{0}$, *has a solution* $\psi \in \Sigma^\ell$.

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Back to spectral Problem

Given L and B in $\mathcal{R}_\ell[D]$

$$LY = \lambda Y \quad , \quad BY = \mu Y \quad , \quad Y = (y_1, \ldots, y_\ell)^t.$$

for

$$L = A_0 + A_1 D \quad \text{and} \quad B = \sum_{j=0}^{n} B_j D^j, \text{ where } n \geq 1 \ ,$$

assuming that A_1 is invertible.

$$\partial\lambda = 0, \ \partial\mu = 0$$

$$P = L - \lambda := L - \lambda I_\ell \text{ and } Q = B - \mu := B - \mu I_\ell$$

Spectral curve

$$P = L - \lambda = (A_0 - \lambda I_\ell) + A_1 D, \quad Q = B - \mu = (B_0 - \mu I_\ell) + \sum_{j=1}^{n} B_j D^j.$$

$$f(\lambda, \mu) = \mathrm{DRes}(L - \lambda, B - \mu) = \det M(L - \lambda, B - \mu)$$

is a polynomial in $K[\lambda, \mu]$

$$f(\lambda, \mu) = (-1)^\ell \mu^\ell + \det(B_n) \det(A_1^{-1})^n \lambda^{n\ell} + q(\lambda, \mu),$$

Theorem B. *If L and B commute then*

$f(\lambda, \mu)$ is a polynomial in $C[\lambda, \mu]$.

The spectral curve of the pair L, B.

$$\Gamma = \{(\lambda, \mu) \in C^2 \mid f(\lambda, \mu) = 0\}.$$

Coupled spectral problem

Apply Theorem A to $P = L - \lambda$ and $Q = B - \mu$ whose matrix coefficients have entries in $\mathcal{F} = K(\lambda, \mu)$.

$\overline{\mathcal{F}}$ algebraic closure, $\mathcal{C} = \overline{\mathcal{C}}$ its field of constants

$\mathcal{E}$ Picard-Vessiot extension of $\overline{\mathcal{F}}$ for

$$DY = N_\lambda Y \quad \text{with} \quad N_\lambda = -A_1^{-1}(A_0 - \lambda I_\ell) \in M_\ell(\overline{\mathcal{F}}).$$

equivalent to

$$(L - \lambda)Y = 0 \text{ with } L = A_0 + A_1 D$$

Proof of Theorem B.

We consider Ψ_λ a fundamental matrix satisfying $(L-\lambda)Y=0$. Then

$$(B-\mu)(\Psi_\lambda)=\Psi_\lambda\cdot\Delta\ ,$$

for some matrix Δ with entries in $\mathcal{C}$. On the other hand,

$$(B-\mu)(\Psi_\lambda)=M(L-\lambda,B-\mu)\Psi_\lambda.$$

thus

$$\mathrm{DRes}(L-\lambda,B-\mu)=\det M(L-\lambda,B-\mu)=\det(\Delta),$$

is a polynomial in

$$K[\lambda,\mu]\cap\mathcal{C}=C[\lambda,\mu].$$

□

Corollary. *Let* $P = (\lambda_0, \mu_0) \in C^2$. *The spectral problem*

$$LY = \lambda_0 Y \quad , \quad BY = \mu_0 Y \ .$$

has a nontrivial solution if and only if $f(P) = 0$,

P *is a point on the spectral curve* Γ *defined by*

$$f(\lambda, \mu) = \mathrm{DRes}(L - \lambda, B - \mu).$$

A common solution ψ *belongs to* ${\Sigma_0}^{\ell}$, *where* Σ_0 *is a Picard-Vessiot extension for the linear differential system*

$$DY = N_{\lambda_0} Y \quad \mathit{with} \;\; N_{\lambda_0} = -A_1^{-1}(A_0 - \lambda_0 I_\ell).$$

Back to AKNS

$$L = \imath \begin{bmatrix} D & u \\ v & -D \end{bmatrix} = A_0 + A_1 D,$$

$$B = \imath \begin{bmatrix} -2D^2 - uv & -2uD - u_x \\ -2vD - v_x & 2D^2 + uv \end{bmatrix} = B_0 + B_1 D + B_2 D^2 \ ,$$

u and v are solutions of the stationary AKNS system, complexified non-linear stationary Schrödinger (NLS) system where v is the complex conjugate of u,

$$u'' + 2u^2 v = 0 \quad , \quad v'' + 2v^2 u = 0 \ . \tag{7}$$

L and B commute. Zero order operator

$$[L, B] = \begin{bmatrix} 0 & -u'' - 2u^2 v \\ v'' + 2v^2 u & 0 \end{bmatrix}, \tag{8}$$

Back to AKNS

$$N_\lambda = -A_1^{-1}(A_0 - \lambda I_2)$$

$$M(L-\lambda, B-\mu) = B_0 - \mu I_2 + B_1 N_\lambda + B_2(N_\lambda^2 + N_\lambda') =$$

$$= \begin{bmatrix} -\imath uv + 2\,\imath\lambda^2 - \mu & \imath u' + 2\,u\lambda \\ \imath v' - 2\,v\lambda & \imath uv - 2\,\imath\lambda^2 - \mu \end{bmatrix}.$$

$$f(\lambda,\mu) = \mathrm{DRes}(L-\lambda, B-\mu) = \mu^2 + 4\,\lambda^4 + I_0\lambda + I_1 \qquad (9)$$

$I_0 = u^2v^2 + v'u'$ and $I_1 = -2\,iv'u + 2\,iu'v$ first integrals of the NLS equation,

$$I_0' = 2uu'v^2 + 2u^2vv' + v''u' + vu'' = 0\ ,\ \ I_1' = -2\imath v''u + 2\imath u''v = 0.$$

(9) defines the spectral curve Γ in $\mathbb{C}^2$.

Example 1: Irreducible curve
$K=\mathbb{C}(e^{2\imath x})$ and NLS potentials $u(x)=e^{-2\imath x}$, $v(x)=2e^{2\imath x}$,

$$L=\imath\begin{bmatrix} D & e^{-2\imath x}\\ 2e^{2\imath x} & -D\end{bmatrix},\ B=\imath\begin{bmatrix} -2D^2-2 & -2e^{-2\imath x}D+2\imath e^{-2\imath x}\\ -4e^{2\imath x}D-4\imath e^{2\imath x} & 2D^2+2\end{bmatrix}$$

The spectral curve Γ is an irreducible singular curve defined by

$$f(\lambda,\mu)=\mu^2+4(\lambda+1)^2(\lambda^2-2\lambda+3)=0$$

Example 2: Reducible curve
$K=\mathbb{C}(x)$ and NLS potentials $u(x)=x$ and $v(x)=0$.
The spectral curve Γ defined by

$$f(\lambda,\mu)=\mu^2+4\lambda^4=0$$

has two irreducible components defined by

$$h_1(\lambda,\mu)=\mu-2i\lambda^2=0 \text{ and } h_2(\lambda,\mu)=\mu+2i\lambda^2=0$$

Contents

BC polynomials

We establish a morphism of rings

$$\rho : C[\lambda, \mu] \longrightarrow C[L, B] := \left\{ \sum a_{i,j} L^i B^j \mid a_{i,j} \in C \right\} \subset \mathcal{R}_\ell[D],$$

defined by $\rho(c) = cI_\ell$, for every $c \in C$,

$$\lambda \mapsto L \text{ and } \mu \mapsto B.$$

Given $g \in C[\lambda, \mu]$

$$g(L, B) := \rho(g)$$

$g \in C[\lambda, \mu]$ is a `Burchnall-Chaundy (BC) polynomial` of the pair L, B if

$$g(L, B) = \mathbf{0}.$$

BC ideal

We call Burchnall-Chaundy (BC) ideal of the pair L, B to the non zero ideal in $C[\lambda, \mu]$ defined as

$$\mathtt{BC}(L, B) = Ker(\rho) = \{g \in C[\lambda, \mu] \mid g(L, B) = \mathbf{0}\}.$$

$$\frac{C[\lambda, \mu]}{\mathtt{BC}(L, B)} \simeq C[L, B].$$

BC ideal

Given commuting MODOs L and B in $\mathcal{R}_\ell[D]$, we assume that L has order 1, with invertible leading coefficient.

$$f(\lambda, \mu) = \mathrm{DRes}(L - \lambda, B - \mu) \text{ in } C[\lambda, \mu]$$

Theorem *Then* $f(L, B)(\Psi_\lambda) = \mathbf{0}$, *for any fundamental matrix* Ψ_λ *of the system* $LY = \lambda Y$.

(K, ∂), constants $C = \overline{C}$ of char 0:

Conjecture $f(L, B) = \mathbf{0}$.

It holds in AKNS and [Grinevich] with coefficients in $M_\ell(\mathbb{C}\{x\})$, $\mathbb{C}\{x\}$ ring of convergent power series.

Classification of algebras

Decomposition of

$$f(\lambda, \mu) = \mathrm{DRes}(L - \lambda, B - \mu)$$

in irreducible factors

$$f = h_1^{\sigma_1} \cdots h_s^{\sigma_s}$$

Theorem C. *Let us assume $f(L, B) = \mathbf{0}$. There exists a polynomial $F = h_1^{r_1} \cdots h_s^{r_s}$ that divides f such that $\mathtt{BC}(L, B) = (F)$. Furthermore*

$$C[L, B] \simeq \frac{C[\lambda, \mu]}{(h_1^{r_1})} \times \cdots \times \frac{C[\lambda, \mu]}{(h_s^{r_s})},$$

whose ring structure is componentwise addition and multiplication.

Algorithm BC-generator

Given commuting MODOs L and B in $\mathcal{R}_\ell[D]$, with L of order one and invertible leading coefficient, return a polynomial $F \in C[\lambda, \mu]$

$$\mathtt{BC}(L, B) = (F).$$

1. *Compute the differential resultant*

$$f(\lambda, \mu) = \mathrm{DRes}(L - \lambda, B - \mu)$$

2. *If $f(L, B) = \mathbf{0}$ then factor f to obtain $h_1^{\sigma_1} \cdots h_s^{\sigma_s}$, each h_i irreducible in $C[\lambda, \mu]$.*
3. *For each $i = 1, \ldots, s$, compute the minimal integer r_i, with $0 \le r_i \le \sigma_i$, such that*

$$\prod_i h_i(L, B)^{r_i} = \mathbf{0}.$$

4. *Return $F = h_1^{r_1} \cdots h_s^{r_s}$.*

$\ell = 2$

Theorem *Let us consider commuting MODOs L and B in $\mathcal{R}_2[D]$, with L of order one and invertible leading coefficient. If $B \notin C[L]$ and $f(L, B) = \mathbf{0}$ then*

$$\mathtt{BC}(L, B) = (f).$$

Classification of algebras $C[L, B]$ for MODOs of size $\ell = 2$:

- If f has one irreducible component then $C[L, B] \simeq C[\lambda, \mu]/(f)$;
- If $f = h_1 \cdot h_2$ then $C[L, B] \simeq C[\lambda, \mu]/(h_1) \times C[\lambda, \mu]/(h_2)$.

$$f(L,B)=\mathbf{0} \text{ and } \mathrm{BC}(L,B)=(f)$$

Example 1: Irreducible curve
$K=\mathbb{C}(e^{2\imath x})$ and NLS potentials $u(x)=e^{-2\imath x}$, $v(x)=2e^{2\imath x}$,
The spectral curve Γ is an irreducible singular curve defined by

$$f(\lambda,\mu)=\mu^2+4(\lambda+1)^2(\lambda^2-2\lambda+3)=0$$

Example 2: Reducible curve
$K=\mathbb{C}(x)$ and NLS potentials $u(x)=x$ and $v(x)=0$.
The spectral curve Γ defined by

$$f(\lambda,\mu)=\mu^2+4\lambda^4=h_1h_2=0$$

$h_1(L,B)\neq\mathbf{0}$ and $h_2(L,B)\neq\mathbf{0}$.

$$C[L,B]\simeq C[\lambda,\mu]/(h_1)\times C[\lambda,\mu]/(h_2)$$

$\ell = 2$: Space of common solutions

Let $P = (\lambda_0, \mu_0)$ be on the curve Γ with $\mu_0 \neq 0$. Let Σ_0 be a Picard-Vessiot field for the system $DY = N_{\lambda_0} Y$.
$B - \mu_0$ restricted to the kernel of $L - \lambda_0$ gives

$$(B - \mu_0)\psi = M(L - \lambda_0, B - \mu_0)\psi \ .$$

$$\xi : \Sigma_0^2 \to \Sigma_0^2 \ , \ \xi(\psi) := M(L - \lambda_0, B - \mu_0)\psi,$$

has a nontrivial kernel $\mathcal{L}$,

$$\mathcal{L} = \left\{(\psi_1, \psi_2) \in \Sigma_0^2 \mid \left(-\imath uv + 2\,\imath {\lambda_0}^2 - \mu_0\right)\psi_1 + \left(\imath u' + 2\,u\lambda_0\right)\psi_2 = 0 \ \right\}$$

(rank 1) fiber bundle over Γ,

$$\phi = -\frac{-\imath uv + 2\,\imath {\lambda_0}^2 - \mu_0}{\imath u' + 2\,u\lambda_0} = \frac{\psi_2}{\psi_1}$$

satisfies the Riccati-type equation $\phi' - u\phi^2 - 2\imath\lambda\phi - v = 0$, since

$$\phi' - u\phi^2 - 2\imath\lambda\phi - v = -u \cdot f(\lambda, \mu) = 0 \ .$$

Example 1: Irreducible curve
$K = \mathbb{C}(e^{2\imath x})$ and NLS potentials $u(x) = e^{-2\imath x}$, $v(x) = 2e^{2\imath x}$,
The spectral curve Γ is an irreducible singular curve defined by

$$f(\lambda, \mu) = \mu^2 + 4(\lambda + 1)^2(\lambda^2 - 2\lambda + 3) = 0$$

The common solution of the coupled spectral problem at a nonbranching point P is

$$\Psi = \begin{pmatrix} 1 \\ \phi \end{pmatrix} \quad \text{with } \phi = -\frac{-2\imath + 2\imath\lambda_0^2 - \mu}{2 + 2\lambda_0} \cdot e^{2\imath x} \ .$$

Example 2: Reducible curve
$K = \mathbb{C}(x)$ and NLS potentials $u(x) = x$ and $v(x) = 0$.
The spectral curve Γ defined by

$$f(\lambda, \mu) = \mu^2 + 4\lambda^4 = h_1 h_2 = 0$$

$$\Psi = \begin{pmatrix} 1 \\ \phi \end{pmatrix} \quad \text{with } \phi = -\frac{2i\lambda_0^2 - \mu_0}{i + 2x\lambda_0}.$$

Available at

Previato, E., Rueda, S. L., Zurro, M. A. (2023).
Burchnall-Chaundy polynomials for matrix ODOs and Picard-Vessiot Theory.
arXiv preprint arXiv:2210.02788.
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Thank you

for your attention !