Integrability and limit cycles in polynomial systems of ODEs

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Outline

- Introduction to the center problem
- Limit cycles: Cyclicity and 16th Hilbert problem
- Algorithmic approach to the problems

References:

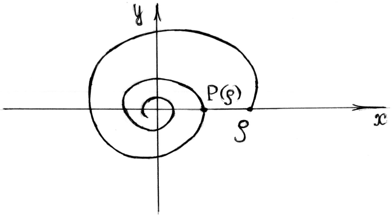
- V. G. Romanovski and D. S. Shafer. *The Center and Cyclicity Problems: A Computational Algebra Approach*. Birkhäuser, Boston, 2009,
- T. Petek and V. G. Romanovski, Computation of Normal Forms for Systems with Many Parameters, arXiv:2305.01739, 2023.

Poincaré center problem

- A center $\Longleftrightarrow$ all solutions near the origin are periodic.
- A focus $\Longleftrightarrow$ all solutions near the origin are spirals.

$$\dot{u} = -v + \sum_{i+j=2}^{n} \alpha_{ij} u^i v^j, \qquad \dot{v} = u + \sum_{i+j=2}^{n} \beta_{ij} u^i v^j . \tag{1}$$

Poincaré return map:



$$\mathcal{P}(\rho) = \rho + \eta_3(\alpha_{ij}, \beta_{ij})\rho^3 + \eta_4(\alpha_{ij}, \beta_{ij})\rho^4 + \dots .$$

Center: $\eta_3 = \eta_4 = \eta_5 = \dots = 0$.

$$\mathcal{P}(\rho) = \rho + \eta_3(\alpha_{ij}, \beta_{ij})\rho^3 + \eta_4(\alpha_{ij}, \beta_{ij})\rho^4 + \dots .$$

Center: $\eta_3 = \eta_4 = \eta_5 = \cdots = 0.$

Poincaré center problem

Find all systems in the family

$$\dot{u} = -v + \sum_{i+j=2}^{n} \alpha_{ij} u^i v^j, \qquad \dot{v} = u + \sum_{i+j=2}^{n} \beta_{ij} u^i v^j,$$

which have a center at the origin.

Bautin ideal: $\mathcal{B} = \langle \eta_3, \eta_4, \dots \rangle \subset \mathbb{R}[\alpha_{ij}, \beta_{ij}]$.

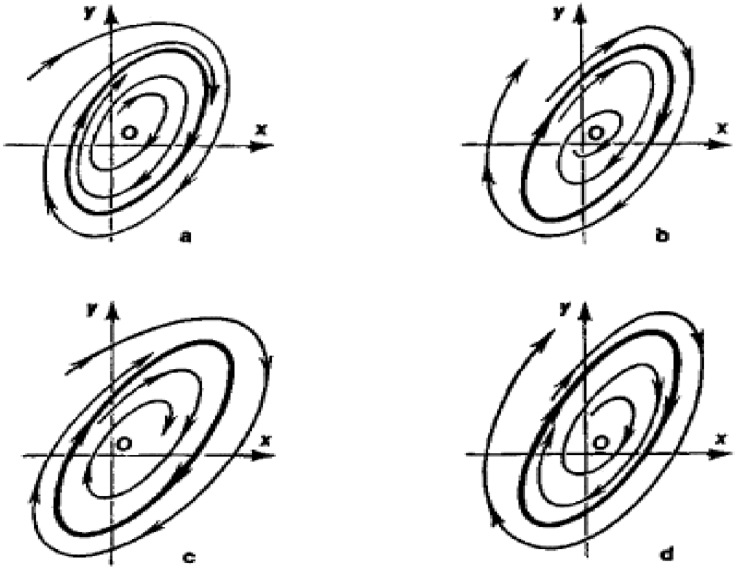
Algebraic counterpart

Find the variety $\mathbf{V}(\mathcal{B})$ of the Bautin ideal $\mathcal{B} = \langle \eta_3, \eta_4, \eta_5 \dots \rangle$.

$$\mathbf{V}(\mathcal{B}) = \{(\alpha_{ij}, \beta_{ij}) \in \mathcal{E} \mid \eta_3(\alpha_{ij}, \beta_{ij}) = \eta_4(\alpha_{ij}, \beta_{ij}) = \cdots = 0\}$$

- $\mathbf{V}(\mathcal{B})$ is called the center variety.

Limit cycles



$$\dot{x} = P_n(x, y), \quad \dot{y} = Q_n(x, y), \qquad (A)$$

$P_n(x, y)$, $Q_n(x, y)$, are polynomials of degree n.
Let $h(P_n, Q_n)$ be the number of limit cycles of system (A) and let $H(n) = \sup h(P_n, Q_n)$.
The question of the second part of the 16th Hilbert's problem:

- find a bound for $H(n)$ as a function of n.

The problem is still unresolved even for $n = 2$.

$n = 2$

- I. Petrovskii, E. Landis, On the number of limit cycles of the equation dy/dx = P(x,y)/Q(x,y), where P and Q are polynomials of 2nd degree (Russian), Mat. Sb. N.S. 37(79) (1955), 209-250
- I. Petrovskii, E. Landis, On the number of limit cycles of the equation dy/dx = P(x,y)/Q(x,y), where P and Q are polynomials (Russian), Mat. Sb. N.S. 85 (1957), 149-168

Song Ling Shi, A concrete example of the existence of four limit cycles for plane quadratic systems, Sci. Sinica 23 (1980), 153-158

- A simpler problem: is $H(n)$ finite? Unresolved.
- An even simpler problem: is $h(P_n, Q_n, a^*, b^*)$ finite?
- H. Dulac, Sur les cycles limite, Bull. Soc. Math. France 51 (1923), 45-188

Around 1980 Yu. Ilyashenko found a mistake in Dulac's proof.

- Chicone and Shafer (1983) proved that for $n = 2$ a fixed system (A) has only finite number of limit cycles in any bounded region of the phase plane.
- Bamòn (1986) and V. R (1986) proved that $h(P_2, Q_2, a^*, b^*)$ is finite.
- Il'yashenko (1991) and Ecalle (1992): $h(P_n, Q_n, a^*, b^*)$ is finite for any n.

Cyclicity and Bautin's theorem

$$\dot{u} = -v + \sum_{j+l=2}^{n} \alpha_{jl} u^j v^l, \quad \dot{v} = u + \sum_{j+l=2}^{n} \beta_{jl} u^j v^l \qquad (2)$$

Poincare map:

$$\mathcal{P}(\rho) = \rho + \eta_2(\alpha_{ij}, \beta_{ij})\rho^2 + \eta_3(\alpha_{ij}, \beta_{ij})\rho^3 + \cdots + \eta_k(\alpha_{ij}, \beta_{ij})\rho^k + \ldots.$$

Let $\mathcal{B} = \langle \eta_3, \eta_4, \ldots \rangle \subset \mathbb{R}[\alpha_{ij}, \beta_{ij}]$ be the ideal generated by all focus quantities η_i. There is k such that

$$\mathcal{B} = \langle \eta_{u_1}, \eta_{u_2}, \ldots, \eta_{u_k} \rangle.$$

The Bautin ideal and Bautin's theorem

Then for any s

$$\eta_s = \eta_{u_1}\theta_1^{(s)} + \eta_{u_2}\theta_2^{(s)} + \cdots + \eta_{u_k}\theta_k^{(k)},$$

$$\mathcal{P}(\rho) - \rho = \eta_{u_1}(1 + \mu_1\rho + \dots)\rho^{u_1} + \cdots + \eta_{u_k}(1 + \mu_k\rho + \dots)\rho^{u_k}.$$

Bautin's Theorem

If $\mathcal{B} = \langle \eta_{u_1}, \eta_{u_2}, \dots, \eta_{u_k} \rangle$ then the cyclicity of system (2) (i.e. the maximal number of limit cycles which appear from the origin after small perturbations) is less or equal to k.

Proof. Bautin N.N. Mat. Sb. (1952) v.30, 181-196 (Russian);
Trans. Amer. Math. Soc. (1954) v.100
Roussarie R. Bifurcations of planar vector fields and Hilbert's 16th problem (1998), Birkhauser.

The cyclicity problem

Find an upper bound for the maximal number of limit cycles in a neighborhood of a center or a focus

By Bautin's theorem:

Algebraic counterpart

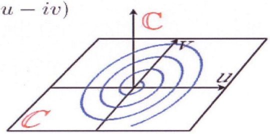
Find a basis for the Bautin ideal $\langle \eta_3, \eta_4, \eta_5, \ldots \rangle$ generated by all coefficients of the Poincaré map

Complexification

Complexification: $x = u + iv \quad (\bar{x} = u - iv)$

$$\dot{x} = \; i(x - \sum_{p+q=1}^{n-1} a_{pq} x^{p+1} \bar{x}^q)$$

$$\dot{\bar{x}} = -\, i(\bar{x} - \sum_{p+q=1}^{n-1} \bar{a}_{pq} \bar{x}^{p+1} x^q)$$



$$\dot{x} = i(x - \sum_{p+q=1}^{n-1} a_{pq} x^{p+1} y^q), \quad \dot{y} = -i(y - \sum_{p+q=1}^{n-1} b_{qp} x^q y^{p+1}) \quad (3)$$

The change of time $d\tau = idt$ transforms (3) to the system

$$\dot{x} = (x - \sum_{p+q=1}^{n-1} a_{pq} x^{p+1} y^q), \; \dot{y} = -(y - \sum_{p+q=1}^{n-1} b_{qp} x^q y^{p+1}). \quad (4)$$

Poincaré-Lyapunov Theorem

The system

$$\frac{du}{dt} = -v + \sum_{i+j=2}^{n} \alpha_{ij} u^i v^j, \quad \frac{dv}{dt} = u + \sum_{i+j=2}^{n} \beta_{ij} u^i v^j \tag{5}$$

has a center at the origin if and only if it admits a first integral of the form

$$\Phi = u^2 + v^2 + \sum_{k+l \geq 2} \phi_{kl} u^k v^l.$$

- Center $\Longleftrightarrow$ local analytic integrability
- A similar approach can be applied to the problem of local integrability for higher dimensional vector field

Local integrability of complex systems

For system

$$\dot{x} = (x - \sum_{p+q=1}^{n-1} a_{pq} x^{p+1} y^q) = P, \;\; \dot{y} = -(y - \sum_{p+q=1}^{n-1} b_{qp} x^q y^{p+1}) = Q, \quad (6)$$

look for a function

$$\Phi(x, y; a_{10}, b_{10}, \ldots) = xy + \sum_{s=3}^{\infty} \sum_{j=0}^{s} v_{j,s-j} x^j y^{s-j}$$

such that

$$\frac{\partial \Phi}{\partial x} P + \frac{\partial \Phi}{\partial y} Q = g_{11}(xy)^2 + g_{22}(xy)^3 + \cdots, \qquad (7)$$

and $g_{11}, g_{22}, \ldots$ are polynomials in a_{pq}, b_{qp}. These polynomials are called *focus quantities*.

- The ideal $\mathcal{B} = \langle g_{11}, g_{22}, \ldots \rangle$ is called the *Bautin ideal*.
- Systems from $\mathbf{V}(\mathcal{B})$ are integrable.

Local Integrability Problem

Find the variety $\mathbf{V}(\mathcal{B})$ of the Bautin ideal $\mathcal{B} = \langle g_{11}, g_{22}, g_{33} \ldots \rangle$.

- Compute polynomials g_{ss} until the chain of varieties (considering as complex varieties) $V(\mathcal{B}_1) \supseteq V(\mathcal{B}_2) \supseteq V(\mathcal{B}_3) \supseteq \ldots$ stabilizes (here $\mathcal{B}_k = \langle g_{11}, \ldots, g_{kk} \rangle$), that is, until we find k_0 such that $V(\mathcal{B}_{k_0}) = V(\mathcal{B}_{k_0+1})$.
- **Show that $V(\mathcal{B}_{k_0}) = V(\mathcal{B})$, that is, that each systems from $V(\mathcal{B}_{k_0})$ admits a first integral of the form (7).**

The center problem is solved for:

• quadratic system: $\dot{x} = x + P_2(x, y), \quad \dot{y} = -y + Q_2(x, y)$
by Dulac (1908) (by Kapteyn (1912) for real systems)

• the linear center perturbed by 3rd degree homogeneous polynomials:
$\dot{x} = x + P_3(x, y), \quad \dot{y} = -y + Q_3(x, y)$
by Sadovski (1974) (by Malkin (1964) for real systems)

• for some particular subfamilies of the cubic system
$\dot{x} = x + P_2(x, y) + P_3(x, y), \quad \dot{y} = -y + Q_2(x, y) + Q_3(x, y)$

• for Lotka-Volterra quartic systems with homogeneous nonlinearities
$\dot{x} = x + xP_3(x, y), \quad \dot{y} = -y + yQ_3(x, y)$
by B. Ferčec, J. Giné, Y. Liu and V. R. (2013)

• for Lotka-Volterra quintic systems with homogeneous nonlinearities
$\dot{x} = x + xP_4(x, y), \quad \dot{y} = -y + yQ_4(x, y)$
by J. Giné and V. R. (2010)

The center variety of the quadratic system

$$\dot{x} = x - a_{10}x^2 - a_{01}xy - a_{-12}y^2, \ \dot{y} = -(y - b_{10}xy - b_{01}y^2 - b_{2,-1}x^2). \quad (8)$$

Theorem (H. Dulac 1908 – C. Christopher & C. Rouseeau, 2001)

The variety of the Bautin ideal of system (8) coincides with the variety of the ideal $\mathcal{B}_3 = \langle g_{11}, g_{22}, g_{33} \rangle$ and consists of four irreducible components:
1) $\mathbf{V}(J_1)$, where $J_1 = \langle 2a_{10} - b_{10}, 2b_{01} - a_{01} \rangle$,
2) $\mathbf{V}(J_2)$, where $J_2 = \langle a_{01}, b_{10} \rangle$,
3) $\mathbf{V}(J_3)$, where $J_3 = \langle 2a_{01} + b_{01}, a_{10} + 2b_{10}, \ a_{01}b_{10} - a_{-12}b_{2,-1} \rangle$,
4) $\mathbf{V}(J_4) = \langle f_1, f_2, f_3, f_4, f_5 \rangle$, where
$f_1 = a_{01}^3 b_{2,-1} - a_{-12}b_{10}^3, \ f_2 = a_{10}a_{01} - b_{01}b_{10}, \ f_3 = a_{10}^3 a_{-12} - b_{2,-1}b_{01}^3,$
$f_4 = a_{10}a_{-12}b_{10}^2 - a_{01}^2 b_{2,-1}b_{01}, \ f_5 = a_{10}^2 a_{-12}b_{10} - a_{01}b_{2,-1}b_{01}^2.$

The cyclicity of the quadratic system

Generalized Bautin's theorem (V. R. & D. Shafer, 2009)

If the ideal $\mathcal{B}$ of all focus quantities of system

$$\dot{x} = (x - \sum_{p+q=1}^{n-1} a_{pq} x^{p+1} y^q), \quad \dot{y} = -(y - \sum_{p+q=1}^{n-1} b_{qp} x^q y^{p+1})$$

is generated by the m first focus quantities,
$\mathcal{B} = \langle g_{11}, g_{22}, \ldots, g_{mm} \rangle$, then at most m limit cycles bifurcate from the origin of the corresponding real system

$$\dot{u} = \lambda u - v + \sum_{j+l=2}^{n} \alpha_{jl} u^j v^l, \quad \dot{v} = u + \lambda v + \sum_{j+l=2}^{n} \beta_{jl} u^j v^l,$$

that is the cyclicity of the system is less or equal to m.

The problem has been solved for:

- The quadratic system ($\dot{x} = P_n, \ \dot{y} = Q_n, \ n = 2$) - Bautin (1952) (Żołądek (1994), Yakovenko (1995), Françoise and Yomdin (1997), Han, Zhang & Zhang (2007)).
- The system with homogeneous cubic nonlinearities - Sibirsky (1965) (Żołądek (1994))

Bautin's theorem for the quadratic system

The cyclicity of the origin of system

$$\dot{u} = \lambda u - v + \alpha_{20} u^2 + \alpha_{11} uv + \alpha_{02} v^2, \ \ \dot{v} = u + \lambda v + \beta_{20} u^2 + \beta_{11} uv + \beta_{02} v^2$$

equals three.

Proof. (V. R., 2007) We have for all k

$$g_{kk}|_{\mathbf{V}(\mathcal{B}_3)} \equiv 0 \tag{9}$$

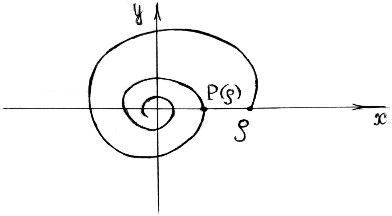
where $\mathcal{B}_3 = \langle g_{11}, g_{22}, g_{33} \rangle$.
Hence, if $\mathcal{B}_3$ is a radical ideal then (9) and Hilbert Nullstellensatz yield that $g_{kk} \in \mathcal{B}_3$. Thus, to prove that an upper bound for the cyclicity is equal to three it is sufficient to show that $\mathcal{B}_3$ is a radical ideal.
With help of SINGULAR we check that

$$std(radical(\mathcal{B}_3)) = std(\mathcal{B}_3). \tag{10}$$

Hence, $\mathcal{B}_3 = \mathcal{B}$. This completes the proof.

$$\dot{u} = -v + \sum_{i+j=2}^{n} \alpha_{ij} u^i v^j, \qquad \dot{v} \ = u + \sum_{i+j=2}^{n} \beta_{ij} u^i v^j \,.$$

Poincaré return map:



$$\mathcal{P}(\rho) = \rho + \eta_3(\alpha_{ij}, \beta_{ij})\rho^3 + \eta_4(\alpha_{ij}, \beta_{ij})\rho^4 + \dots .$$

Center: $\eta_3 = \eta_4 = \eta_5 = \cdots = 0.$

$\mathcal{B} = \langle \eta_3, \eta_4, \eta_5, \dots \rangle$, $\mathbf{V}(\mathcal{B}) = V_1 \cup \dots V_m$.

• Roughly speaking the number of small limit cycles in bifurcating in a neighborhood of V_k is equal to the codimension of V_k in the space of parameters.

Computation of necessary conditions of integrability

$$\dot{x} = \quad x - \sum_{(p,q)\in S} a_{pq}x^{p+1}y^q = x(1 - \sum_{(p,q)\in S} a_{pq}x^p y^q) = x(1 - \tilde{P}(x,y)) = P(x,y),$$
$$\dot{y} = \quad -y + \sum_{(p,q)\in S} b_{qp}x^q y^{p+1} = y(-1 - \sum_{(p,q)\in S} b_{qp}x^q y^p) = y(-1 - \tilde{Q}(x,y)) = Q(x,y), \tag{11}$$

$S \subset \mathbb{N}_{-1} \times \mathbb{N}_0$, $\mathbb{N}_{-1} = \{0 \cup -1\} \cup \mathbb{N}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. $\mathcal{F}$ – the space of formal power series

$$\Psi = \psi_{11}xy + \sum_{k_1+k_2>2} \psi_{k_1k_2}x^{k_1}y^{k_2}.$$

$$D : \mathcal{F} \to \mathcal{F}, \quad D := \frac{\partial \Psi}{\partial x}P(x,y) + \frac{\partial \Psi}{\partial y}Q(x,y) = D^{(1)} + D^{(2)}$$

$$D^{(1)} := x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} \qquad D^{(2)} := -x\tilde{P}\frac{\partial}{\partial x} - y\tilde{Q}\frac{\partial}{\partial y}.$$

$$\mathcal{F} = \mathrm{im}\, D^{(1)} \oplus \ker D^{(1)}.$$

Problem. Find $\Psi \in \mathcal{F}$ such that $D(\Psi) = G \in \ker D^{(1)}$.

In such case is said that a $\Psi \in \mathcal{F}$ is in a *normal form* with respect to (11).

Problem. Find $\Psi \in \mathcal{F}$ such that

$$D(\Psi) = G \in \ker D^{(1)}.$$

• $\mathcal{F}_j$ $(j \geq 2)$ – the space of homogeneous polynomials of degree j.

$$\Phi_j = \sum_{k_1+k_2=j} \phi_{k_1 k_2} x^{k_1} y^{k_2}.$$

The restriction of D to $\mathcal{F}_j$:

$$D_j : \mathcal{F}_j \to \mathcal{F}_j$$

$D_j^{(1)}$ is semisimple: eigenvalues $\beta_{k_1 k_2} = k_1 - k_2$, eigenvectors $x^{k_1} y^{k_2}$.

$$\mathcal{F}_j = \mathtt{im}\, D_j^{(1)} \oplus \ker D_j^{(1)}.$$

Let $\mathcal{P}_j$ be the projection on $\ker D_j^{(1)}$.

Computation of a formal series in the normal form

- $\Psi := xy, \qquad G := 0.$
- Assume that equation Ψ is in the normal form to order $j-1$. Let $g_j = \mathcal{P}_j(D(\Psi)),\ h_j = (I - \mathcal{P}_j)(D(\Psi))$.
  Solve
  $$D_j \Phi_j = h_j + g_j. \tag{12}$$
- $\Psi = \Psi + \Phi_j, \qquad G = G + g_j.$

A grading of the formal series module

$$\dot{x} = \quad x - \sum_{(p,q)\in S} a_{pq} x^{p+1} y^q = x(1 - \sum_{(p,q)\in S} a_{pq} x^p y^q) = P(x,y),$$

$$\dot{y} = \quad -y + \sum_{(p,q)\in S} b_{qp} x^q y^{p+1} = -y(1 - \sum_{(p,q)\in S} b_{qp} x^q y^p) = Q(x,y),$$

ℓ – the number of parameters of each equation.
$S = \{a_{p_1,q_1}, a_{p_2,q_2}, \dots, a_{p_\ell,q_\ell}, b_{q_\ell,p_\ell}, \dots, b_{q_2,p_2}, b_{q_1,p_1}\}$ – the ordered set.
$k[a,b] := k[a_{p_1,q_1}, a_{p_2,q_2}, \dots, a_{p_\ell,q_\ell}, b_{q_\ell,p_\ell}, \dots, b_{q_2,p_2}, b_{q_1,p_1}]$.
Any monomial appearing in $k[a,b]$ has the form

$$a_{p_1,q_1}^{\nu_1} a_{p_2,q_2}^{\nu_2} \cdots a_{p_\ell,q_\ell}^{\nu_\ell} b_{q_\ell,p_\ell}^{\nu_{\ell+1}} \cdots b_{q_2,p_2}^{\nu_{2\ell-1}} b_{q_1,p_1}^{\nu_{2\ell}}.$$

For $\nu \in \mathbb{N}_0^{2\ell}$ we write

$$(ab)^\nu \stackrel{\text{def}}{=} a_{p_1,q_1}^{\nu_1} a_{p_2,q_2}^{\nu_2} \cdots a_{p_\ell,q_\ell}^{\nu_\ell} b_{q_\ell,p_\ell}^{\nu_{\ell+1}} \cdots b_{q_2,p_2}^{\nu_{2\ell-1}} b_{q_1,p_1}^{\nu_{2\ell}}$$

$$L : \mathbb{N}_0^{2\ell} \to \mathbb{Z}^2$$

$$L(\nu) = \begin{pmatrix} p_1 \\ q_1 \end{pmatrix} \nu_1 + \cdots + \begin{pmatrix} p_\ell \\ q_\ell \end{pmatrix} \nu_\ell + \begin{pmatrix} q_\ell \\ p_\ell \end{pmatrix} \nu_{\ell+1} + \cdots + \begin{pmatrix} q_1 \\ p_1 \end{pmatrix} \nu_{2\ell} = \begin{pmatrix} L^1(\nu) \\ L^2(\nu) \end{pmatrix}.$$

$$L(\nu) = \begin{pmatrix} p_1 \\ q_1 \end{pmatrix} \nu_1 + \cdots + \begin{pmatrix} p_\ell \\ q_\ell \end{pmatrix} \nu_\ell + \begin{pmatrix} q_\ell \\ p_\ell \end{pmatrix} \nu_{\ell+1} + \cdots + \begin{pmatrix} q_1 \\ p_1 \end{pmatrix} \nu_{2\ell} = \begin{pmatrix} L^1(\nu) \\ L^2(\nu) \end{pmatrix}.$$

Denote $\mathcal{Q} = \{(k_1, k_2) \in \mathbb{N}_{-1} \times \mathbb{N}_{-1};\ k_1 + k_2 \geq 0\}$.

Definition

For $(i,j) \in \mathcal{Q}$, a (Laurent) polynomial $f \in k[a,b]$, $f = \sum_{\nu \in Supp(f)} f^{(\nu)}(ab)^\nu$, is an (i,j)–polynomial if for every $\nu \in Supp(f)$ $L(\nu) = (i,j)$.

Let $R_{(i,j)}$ will be the subset of $k[a,b]$ consisting of all (i,j)–polynomials,

$$R = \oplus_{(i,j)\in\mathcal{Q}} R_{(i,j)}.$$

$$R_{(j,k)} R_{(s,t)} \subseteq R_{(j+s,k+t)} \quad \Rightarrow R \ \text{ is a graded ring.}$$

Let M be the set of all formal power series of the form

$$\Psi(x,y) = v_{0,0}xy + \sum_{i+j>2} v_{i-1,j-1} x^i y^j = xy \left(v_{0,0} + \sum_{i+j>2} v_{i-1,j-1} x^{i-1} y^{j-1} \right), \tag{13}$$

such that the $v_{i-1,j-1}$ is a $(i-1,j-1)$-polynomial.
Denoting $M_{(i,j)} = R_{(i,j)} x^{i+1} y^{j+1}$ we have

$$M := \oplus_{(i,j)\in\mathcal{Q}} M_{(i,j)}.$$

M is a vector space over $\mathbb{Q}$ and a graded R-module.

Theorem

For system

$$\dot{x} = x - \sum_{(p,q)\in S} a_{pq}x^{p+1}y^q, \;\; \dot{y} = -y + \sum_{(p,q)\in S} b_{qp}x^q y^{p+1},$$

there exists

$$\Psi(x,y) = v_{0,0}xy + \sum_{i+j>2} v_{i-1,j-1}x^i y^j = xy\left(v_{0,0} + \sum_{i+j>2} v_{i-1,j-1}x^{i-1}y^{j-1}\right) \in M$$

such that

(a) $D(\Psi) \in \ker D^{(1)}$, *more precisely,*

$$D(\Psi) = \sum_{k=1}^{\infty} g_{kk}(xy)^{k+1} \; (\forall\; k \geq 1 \; g_{kk} \in R_{kk});$$

(b) *for every pair* $(i,j) \in \mathbb{N}_{-1} \times \mathbb{N}_{-1}$, $i+j \geq 0$, $v_{ij} \in R_{ij}$;

(c) *for every* $k \geq 1$, $v_{k,k} = 0$.

Integrability problem: Compute the variety of $\mathcal{B} = \langle g_{11}, g_{22}, \ldots \rangle$.

Computation of g_{kk}

$$L(\nu) = \binom{p_1}{q_1}\nu_1 + \cdots + \binom{p_\ell}{q_\ell}\nu_\ell + \binom{q_\ell}{p_\ell}\nu_{\ell+1} + \cdots + \binom{q_1}{p_1}\nu_{2\ell} = \binom{L^1(\nu)}{L^2(\nu)}.$$

R is our graded ring of formal power series of 2ℓ variables $a_{p_1,q_1}, a_{p_2,q_2}, \ldots, a_{p_\ell,q_\ell}, b_{q_\ell,p_\ell}, \ldots, b_{q_2,p_2}, b_{q_1,p_1}$ with the coefficients in $\mathbb{Q}$.
$\nu \in \mathbb{N}_0^{2\ell}$, $|\nu| = \nu_1 + \cdots + \nu_{2\ell}$.
For $\Psi \in M$:

$$\Psi(x,y) = v_{0,0}xy + \sum_{i+j>2} v_{i-1,j-1}x^i y^j,$$

define the map

$$\pi : M \to R$$

$$\pi(\Psi(x,y)) = \Psi(1,1).$$

$$V(a,b) = \Psi(1,1) = v_{0,0} + \sum_{i+j>2} v_{i-1,j-1} = \sum_{|\nu|\geq 0} V(\nu)(ab)^\nu.$$

π is an isomorphism with the inverse

$$\pi^{-1}(V) = \sum_{|\nu|\geq 0} V(\nu)(ab)^\nu x^{L_1(\nu)+1} y^{L_2(\nu)+1}.$$

Theorem

There exists $\Psi \in M$, *such that* $D(\Psi) = g(x,y) \in \ker D^{(1)}$, *where*

$$g(x,y) = \sum_{k=1}^{\infty} g_{kk}(xy)^{k+1} \ (\forall\ k \geq 1\ g_{kk} \in R_{kk}),$$

$V(a,b) = \pi(\Psi)$ *and* $G(a,b) = \pi(g(x,y) = \sum_{\nu} G(\nu)(ab)^{\nu}$ *are computed by*

$$(L_1(\nu) - L_2(\nu))V(\nu) = \sum_{j=1}^{\ell} V(\nu_1,\dots,\nu_j - 1,\dots,\nu_{2\ell})(L_1(\nu_1,\dots,\nu_j - 1,\dots,\nu_{2\ell}) + 1)$$
$$- \sum_{j=\ell+1}^{2\ell} V(\nu_1,\dots,\nu_j - 1,\dots,\nu_{2\ell})(L_2(\nu_1,\dots,\nu_j - 1,\dots,\nu_{2\ell}) + 1)$$

$$G(\nu) = \sum_{j=1}^{\ell} V(\nu_1,\dots,\nu_j - 1,\dots,\nu_{2\ell})(L_1(\nu_1,\dots,\nu_j - 1,\dots,\nu_{2\ell}) + 1)$$
$$- \sum_{j=\ell+1}^{2\ell} V(\nu_1,\dots,\nu_j - 1,\dots,\nu_{2\ell})(L_2(\nu_1,\dots,\nu_j - 1,\dots,\nu_{2\ell}) + 1)$$

The equation for V is a difference equation.

$$F(u_1,\dots,u_n)=\sum_\nu F_{(\nu_1,\dots,\nu_n)}u_1^{\nu_1}\dots u_n^{\nu_n}=\sum_\nu F_{(\nu)}u^{(\nu)}$$

$$\sum_\nu \nu_i F_{(\nu;i)}u^{(\nu)}=u_i(u_iF)'_{u_i},\ \sum_\nu \nu_i F_{(\nu;j)}u^{(\nu)}=u_iu_jF'_{u_i}\ (i\neq j),\qquad (14)$$

$$\sum_\nu \nu_i F_{(\nu)}u^{(\nu)}=u_iF'_{u_i},\ \sum_\nu \nu_i F_{(\nu;j)}u^{(\nu)}=u_jF,$$

where

$$F_{(\nu;j)}:=F_{(\nu_1,\dots,\nu_{j-1},\nu_j-1,\nu_{j+1},\dots,\nu_n)}.$$

Denote $|a|=\sum_{(i,j)\in S}a_{ij}$, $|b|=\sum_{(j,i)\in S}b_{ij}$.

$$\mathcal{A}(V)=\sum_{(i,j)\in S}\frac{\partial V}{\partial a_{ij}}a_{ij}(i-j-i|a|+j|b|)+\sum_{(j,i)\in S}\frac{\partial V}{\partial b_{ij}}b_{ij}(i-j-i|a|+j|b|)$$
$$-V(|a|-|b|).\qquad (15)$$

$$\mathcal{A}:R\longrightarrow R$$

$$R=\mathrm{im}\mathcal{A}\oplus\ker\mathcal{A},\qquad \ker\mathcal{A}=\oplus R_{(i,i)}.$$

$$\mathcal{A}(V(a,b)) = G(a,b) = \sum_k g_{kk}(a,b).$$

Problem: Find the variety of the Bautin ideal $B = \langle g_{11}, g_{22}, \ldots \rangle$. Let $B_m = \langle g_{11}, g_{22}, \ldots, g_{mm} \rangle$.

- Compute until find a B_s: $\sqrt{B_1} \subset \sqrt{B_2} \subset \cdots \subset \sqrt{B_{s-1}} = \sqrt{B_s}$.
- Compute decomposition $\sqrt{B_s} = \cap_{i=1}^r P_r$, where P_r are prime. Question. Can the grading of R help to compute the decomposition in a more efficient way?
- Let $P_k = \langle f_1(a,b), \ldots, f_u(a,b) \rangle$. Does the system

$$\begin{cases} \mathcal{A}(V) = 0, \\ f_1(a,b) = \cdots = f_u(a,b) = 0 \end{cases}$$

has a nontrivial solution $V = 1 + \ldots$?

Quadratic system

$$\begin{aligned}\dot{x} &= \quad \left(x - a_{10}x^2 - a_{01}xy - a_{-12}y^2\right)\\ \dot{y} &= -\left(y - b_{2,-1}x^2 - b_{10}xy - b_{01}y^2\right)\,.\end{aligned} \tag{16}$$

$$B_3 = \langle g_{11}, g_{22}, g_{33} \rangle = \cap_{i=1}^4 J_i \,:$$

- $V_1 = \mathbf{V}(J_1)$, where $J_1 = \langle 2a_{10} - b_{10},\ 2b_{01} - a_{01} \rangle$;
- $V_2 = \mathbf{V}(J_2)$, where $J_2 = \langle a_{01},\ b_{10} \rangle$;
- $V_3 = \mathbf{V}(J_3)$, where $J_3 = \langle 2a_{01} + b_{01},\ a_{10} + 2b_{10},\ a_{01}b_{10} - a_{-12}b_{2,-1} \rangle$;
- $V_4 = \mathbf{V}(J_4)$, where $J_4 = \langle f_1, f_2, f_3, f_4, f_5 \rangle$, where $f_1 = a_{01}^3 b_{2,-1} - a_{-12} b_{10}^3$, $f_2 = a_{10}a_{01} - b_{01}b_{10}$, $f_3 = a_{10}^3 a_{-12} - b_{2,-1} b_{01}^3$, $f_4 = a_{10}a_{-12}b_{10}^2 - a_{01}^2 b_{2,-1} b_{01}$, $f_5 = a_{10}^2 a_{-12} b_{10} - a_{01} b_{2,-1} b_{01}^2$.

Hamiltonian system: $J_1 = \langle 2a_{10} - b_{10}, 2b_{01} - a_{01} \rangle \Rightarrow$
$a_{01} = 2b_{01}, \;\; b_{10} = 2a_{10}$

$$\mathcal{A}(V) := a_{01}(|b|-1)\frac{\partial V}{\partial a_{01}} + a_{10}(1-|a|)\frac{\partial V}{\partial a_{10}} + a_{-12}(|a|+2|b|-3)\frac{\partial V}{\partial a_{-12}} +$$
$$b_{01}(|b|-1)\frac{\partial V}{\partial b_{01}} + b_{10}(1-|a|)\frac{\partial V}{\partial b_{10}} + b_{2,-1}(-2|a|-|b|+3)\frac{\partial V}{\partial b_{2,-1}} - V(|a|-|b|).$$

$|a| = a_{10} + a_{01} + a_{-12}, \qquad |b| = b_{01} + b_{10} + b_{2,-1}.$

- $$\mathcal{A}(V) := a_{10}(1-|a|)\frac{\partial V}{\partial a_{10}} + a_{-12}(|a| + 2|b| - 3)\frac{\partial V}{\partial a_{-12}} +$$
$$b_{01}(|b|-1)\frac{\partial V}{\partial b_{01}} + b_{2,-1}(-2|a| - |b| + 3)\frac{\partial V}{\partial b_{2,-1}} - V(|a| - |b|).$$
- $V = 1 - a_{-12}/3 - b_{2,-1}/3 - a_{10} - b_{01}$ is a solution to $\mathcal{A}(V) - V(|a| - |b|) = 0$.
- $$H = -\left(xy - \frac{a_{-12}}{3}y^3 - \frac{b_{2,-1}}{3}x^3 - a_{10}x^2y - b_{01}xy^2\right)$$

$\mathbf{V}(J_3)$, where $J_3 = \langle 2a_{01} + b_{01}, a_{10} + 2b_{10},\ a_{01}b_{10} - a_{-12}b_{2,-1}\rangle$,

$$\begin{cases} \mathcal{A}(V) := a_{01}(|b|-1)\frac{\partial V}{\partial a_{01}} + \ a_{10}(1-|a|)\frac{\partial V}{\partial a_{10}} \ + a_{-12}(|a|+2|b|-3)\frac{\partial V}{\partial a_{-12}} \\ +b_{01}(|b|-1)\frac{\partial V}{\partial b_{01}} + \ b_{10}(1-|a|)\frac{\partial V}{\partial b_{10}} \ + b_{2,-1}(-2|a|-|b|+3)\frac{\partial V}{\partial b_{2,-1}} \\ -V(|a|+|b|) \\ 2a_{01} + b_{01} = a_{10} + 2b_{10} = a_{01}b_{10} - a_{-12}b_{2,-1} = 0 \end{cases}$$

$\Rightarrow$

$$\mathcal{A}(V) := a_{01}(|b|-1)\frac{\partial V}{\partial a_{01}} + b_{10}(1-|a|)\frac{\partial V}{\partial b_{10}} + \\ b_{2,-1}(-|b|-2|a|+3)\frac{\partial V}{\partial b_{2,-1}} - V(|a|+|b|) \quad (17)$$

Darboux integral

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y), \quad x, y \in \mathbb{C} \quad P, Q \text{ are polynomials.} \tag{18}$$

$f(x, y) \in \mathbb{C}[x, y]$ defines algebraic invariant curve $f(x, y) = 0$ of system (18) if $\exists k(x, y) \in \mathbb{C}[x, y]$ such that

$$D(f) := \frac{\partial f}{\partial x} P + \frac{\partial f}{\partial y} Q = kf. \tag{19}$$

$k(x, y)$ is called a *cofactor* of f. Suppose that the curves defined by

$$f_1 = 0, \ldots, f_s = 0$$

are invariant algebraic curves of system (18) with the cofactors $k_1, \ldots, k_s$. If

$$\sum_{j=1}^{s} \alpha_j k_j = 0\,, \tag{20}$$

then $H = f_1^{\alpha_1} \cdots f_s^{\alpha_s}$ is a (Darboux) first integral of the system (18).

- $f = \sum_{i+j=0}^{n} c_{ij} x^i y^j, \quad k = \sum_{i+j=0}^{m-1} d_{ij} x^i y^j$. ($m$ is the degree of the system).
- Equal the coefficients of the same terms in $D(f) = kf$.
- Solve the obtained system of polynomial equations for c_{ij}, d_{ij}.

$$\ell_1 = 1 + 2\, b_{10}\, x - a_{01}\, b_{2,-1}\, x^2 + \; 2\, a_{01}\, y + 2\, a_{01}\, b_{10}\, x\, y - \frac{a_{01}\, b_{10}^2}{b_{2,-1}}\, y^2,$$

$$\ell_2 = (2\, b_{10}\, b_{2,-1}^2 + 6\, b_{10}^2\, b_{2,-1}^2\, x + 3\, {b_{10}}^3\, b_{2,-1}^2\, x^2 - 3\, a_{01}\, b_{10}\, b_{2,-1}^3\, x^2 - a_{01}\, b_{10}^2\, b_{2,-}^3 \cdots$$
$$6\, a_{01}\, b_{10}\, b_{2,-1}^2\, y - 3\, b_{10}^4\, b_{2,-1}\, x\, y + 6\, a_{01}\, b_{10}^2\, b_{2,-1}^2\, x\, y - 3\, a_{01}^2\, b_{2,-1}^3\, x\, y + 3\, a_{01}\, b_{10}^3\, b_{2,-}^2 \cdots$$
$$3\, a_{01}^2\, b_{10}\, b_{2,-1}^3 x^2\, y - 3\, a_{01}\, b_{10}^3\, b_{2,-1}\, y^2 + 3\, a_{01}^2\, b_{10}\, b_{2,-1}^2\, y^2 - 3\, a_{01}\, b_{10}^4\, b_{2,-1}\, x\, y^2 + 3\, a_0^2 \cdots$$
$$a_{01}\, b_{10}^5\, y^3 - a_{01}^2\, b_{10}^3\, b_{2,-1}\, y^3)/(2\, b_{10}\, b_{2,-1}^2)$$

with the cofactors $k_1 = 2\,(b_{10}\, x - a_{01}\, y)$ and $k_2 = 3\,(b_{10}\, x - a_{01}\, y)$.
The equation

$$\alpha_1 k_1 + \alpha_2 k_2 = 0$$

has a solution $\alpha_1 = -3, \alpha_2 = 2, \Longrightarrow$

$$\Psi = \ell_1^{-3} \ell_2^2 \equiv c.$$

$$\ell_1 = 1 + 2\, b_{10}\, x - a_{01}\, b_{2,-1}\, x^2 + \; 2\, a_{01}\, y + 2\, a_{01}\, b_{10}\, x\, y - \frac{a_{01}\, b_{10}^2}{b_{2,-1}}\, y^2,$$

$$\ell_2 = (2\, b_{10}\, b_{2,-1}^2 + 6\, b_{10}^2\, b_{2,-1}^2\, x + 3\, {b_{10}}^3\, b_{2,-1}^2\, x^2 - 3\, a_{01}\, b_{10}\, b_{2,-1}^3\, x^2 - a_{01}\, b_{10}^2\, b_{2,-}^3 \ldots$$
$$6\, a_{01}\, b_{10}\, b_{2,-1}^2\, y - 3\, b_{10}^4\, b_{2,-1}\, x\, y + 6\, a_{01}\, b_{10}^2\, b_{2,-1}^2\, x\, y - 3\, a_{01}^2\, b_{2,-1}^3\, x\, y + 3\, a_{01}\, b_{10}^3\, b_{2,-}^2 \ldots$$
$$3\, a_{01}^2\, b_{10}\, b_{2,-1}^3 x^2\, y - 3\, a_{01}\, b_{10}^3\, b_{2,-1}\, y^2 + 3\, a_{01}^2\, b_{10}\, b_{2,-1}^2\, y^2 - 3\, a_{01}\, b_{10}^4\, b_{2,-1}\, x\, y^2 + 3\, a_0^2 \ldots$$
$$a_{01}\, b_{10}^5\, y^3 - a_{01}^2\, b_{10}^3\, b_{2,-1}\, y^3)/(2\, b_{10}\, b_{2,-1}^2)$$

with the cofactors $k_1 = 2\,(b_{10}\, x - a_{01}\, y)$ and $k_2 = 3\,(b_{10}\, x - a_{01}\, y)$.
First integral

$$\Psi = \ell_1^{-3} \ell_2^2 \equiv c.$$

$$\hat{\mathcal{A}}(V) := a_{01}(|b|-1)\frac{\partial V}{\partial a_{01}} + b_{10}(1-|a|)\frac{\partial V}{\partial b_{10}} + b_{2,-1}(-|b|-2|a|+3)\frac{\partial V}{\partial b_{2,-1}}$$

($\hat{\mathcal{A}}$ is $\mathcal{A}$ without $-V(|a|+|b|)$)
$L_1 = 1 + 2a_{01} + 2b_{10} + 2a_{01}b_{10} - (a_{01}b_{10}^2)/b_{2,-1} - a_{01}b_{2,-1}$
$L_2 = (a_{01}b_{10}^5 - 3a_{01}b_{10}^3 b_{2,-1} - a_{01}^2 b_{10}^3 b_{2,-1} - 3b_{10}^4 b_{2,-1} - 3a_{01}b_{10}^4 b_{2,-1} + 2b_{10}b_{2,-1}^2 + 6a_{01}b_{10}b_{2,-1}^2 + 3a_{01}^2 b_{10}b_{2,-1}^2 + 6b_{10}^2 b_{2,-1}^2 + 6a_{01}b_{10}^2 b_{2,-1}^2 + 3a_{01}^2 b_{10}^2 b_{2,-1}^2 + 3b_{10}^3 b_{2,-1}^2 + 3a_{01}b_{10}^3 b_{2,-1}^2 - 3a_{01}^2 b_{2,-1}^3 - 3a_{01}b_{10}b_{2,-1}^3 - 3a_{01}^2 b_{10}b_{2,-1}^3 - a_{01}b_{10}^2 b_{2,-1}^3 + a_{01}^2 b_{2,-1}^4)/(2b_{10}b_{2,-1}^2))$

$$\hat{\mathcal{A}}(L_1) = K_1 L_1, \quad \hat{\mathcal{A}}(L_2) = K_2 L_2,$$

$K_1 = -2(a_{01} - b_{10}), \quad K_1 = -3(a_{01} - b_{10}),$
$U = L_1^{-3} L_2^2$ is a solution to $\hat{\mathcal{A}}(U) = 0$

$$V = \frac{U-1}{-6a_{01}b_{10} - (3b_{10}^3)/b_{2,-1} - (3a_{01}^2 b_{2,-1})/b_{10})}$$

is a solution to

$$\hat{\mathcal{A}}(V) = V(|a|+|b|).$$

$$J_4 = \langle a_{01}^3 b_{2,-1} - a_{-12} b_{10}^3, a_{10} a_{01} - b_{01} b_{10}, a_{10}^3 a_{-12} - b_{2,-1} b_{01}^3, a_{10} a_{-12} b_{10}^2 - a_{01}^2 b_{2,-1} b_{01}, a_{10}^2 a_{-12} b_{10} - a_{01} b_{2,-1} b_{01}^2 \rangle.$$

$$\begin{cases} \mathcal{A}(V) := a_{01}(|b|-1)\frac{\partial V}{\partial a_{01}} + \ a_{10}(1-|a|)\frac{\partial V}{\partial a_{10}} \ + a_{-12}(|a|+2|b|-3)\frac{\partial V}{\partial a_{-12}} \\ +b_{01}(|b|-1)\frac{\partial V}{\partial b_{01}} + \ b_{10}(1-|a|)\frac{\partial V}{\partial b_{10}} \ + b_{2,-1}(-2|a|-|b|+3)\frac{\partial V}{\partial b_{2,-1}} \\ -V(|a|+|b|) \\ |a| = a_{10} + a_{01} + a_{-12}, \qquad |b| = b_{01} + b_{10} + b_{2,-1}. \\ a_{01}^3 b_{2,-1} - a_{-12} b_{10}^3 = a_{10} a_{01} - b_{01} b_{10} = a_{10}^3 a_{-12} - b_{2,-1} b_{01}^3 = \\ a_{10} a_{-12} b_{10}^2 - a_{01}^2 b_{2,-1} b_{01} = a_{10}^2 a_{-12} b_{10} - a_{01} b_{2,-1} b_{01}^2 = 0. \end{cases}$$

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Thank you for your attention!