Chained and Triangular Flat Systems Illustrated by the Aircraft Example

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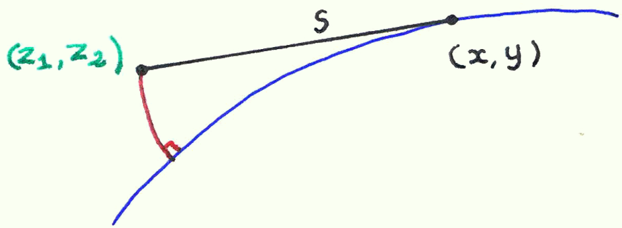
Plan of the talk

FLAT SYSTEMS

Examples… Monge – Petitot’s wheel



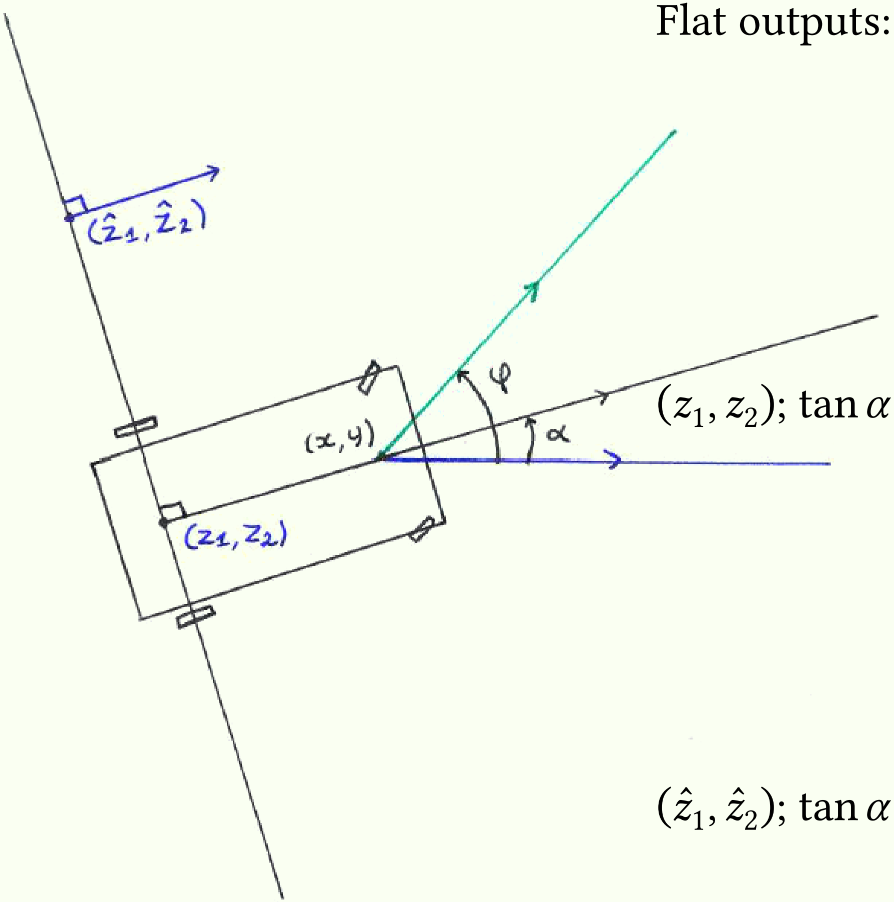
$$\begin{aligned} s' &= u \\ x' &= \cos\alpha\, u \\ y' &= \sin\alpha\, u \\ \alpha' &= v \end{aligned}$$

Linearizing outputs: $(z_1 := x - \cos\alpha\, s, z_2 := y - \sin\alpha\, s)$.

$$\tan\alpha = \frac{-z_1'}{z_2'}$$

$$z_1' = \sin\alpha\, \alpha'\, s$$

Car



Flat outputs:

(z_1, z_2); $\tan\alpha = \frac{z_2'}{z_1'}$.

$(\hat{z}_1, \hat{z}_2)$; $\tan\alpha = \frac{\hat{z}_2'}{\hat{z}_1'}$.

Mathematical definitions

Fliess, Lévine, Martin and Rouchon, 1991.

Differential algebra

Definition. — A differential field extension $\mathscr{G}/\mathscr{F}$ is *flat* if $\overline{\mathscr{G}/\mathscr{F}}$ is isomorphic to $\overline{\mathscr{F}\langle z_1, \ldots, z_m\rangle}$.
The generators of the differentialy transcendental extension z_1, ...z_m are *flat outputs*.

Some flat systems do not admit algebraic flat outputs.

$x' = u$, $y' = yu + 1$; flat output: $z = e^{-x}y$;
parametrization $x = \ln(z')$, $y = z/z'$.

Diffiety theory (Vinogradov)

Definition. — A *diffiety* is a manifold of denumerable dimension, equipped with a derivation or a vector field: the *Cartan field* δ (that we will denote d_t).

The function on the manifold are $\mathscr{C}^\infty$ and only depend on a *finite number* of derivatives.

A morphism $\phi : U_{\delta_1} \mapsto V_{\delta_2}$ is such that $\delta_1 \circ \phi^* = \phi^* \circ \delta_2$. (Lie-Bäcklund equivalence)

On may glue local charts to build new diffieties.

Examples

$\mathbf{T}^m$ is the *trivial diffiety*: $(\mathbf{R}^m)^\infty$ with derivation

$$\delta := \sum_{i=1}^{m} \sum_{k\in\mathbf{N}} u_j^{(k+1)} \frac{\partial}{\partial u_j^{(k)}}.$$

$\mathbf{R}_{\partial_t}$ where ∂_t denote ∂/∂_t.

The jet space $\mathbf{J}(\mathbf{R}, \mathbf{R}^m)$ is isomorphic to $\mathbf{R}_{\partial_t} \times \mathbf{T}^m$, with derivation denoted by d_t.

Definition. — A diffiety is flat if it contains a dense open set W such that any point $\eta \in W$ admits a neighborhood isomorphic to an open set of the jet space $\mathbf{J}(\mathbf{R}, \mathbf{R}^m)$. Such points are *flat points*.

Points around which no such isomorphism exists are *flat singularities*. Kaminski, Lévine and FO, 2018.

Local flatness necessary condition

We consider a system $x_i' = f_i(x,u)$.

Theorem. — *If a point (ξ,v) is flat, then the $\mathbf{R}[\![t]\!]\mathrm{d}_t$-module defined by the linearized system at (ξ,v):*

$$\mathrm{d}x_i' = \sum_{i=1}^{n} \mathrm{j}_{(\xi,v)} \frac{\partial f}{\partial x_i} \mathrm{d}x_i + \sum_{j=1}^{m} \mathrm{j}_{(\xi,v)} \frac{\partial f}{\partial u_j} \mathrm{d}u_j$$

is free.

Proof. — If $z_i = Z_i(x,u,u',\ldots)$ are flat outputs defined and regular at (ξ,v), then $\mathrm{d}Z_i$ is a basis of the module.

Kaminski, Lévine and FO 2020.

Sluis and Rouchon's criterion

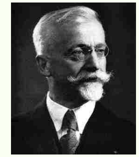
Sluis 1993, Rouchon 1994.
Let $D := \sum_{i=1}^{n} C_i \frac{\partial}{\partial x_i'}$

THEOREM. — If a system such that $P(x_1', \ldots, x_n', x_1, \ldots, x_n) = 0$ defines a flat diffiety in some open space, then the homogeneous ideal $(D^k P | k \in \mathbf{N})$ admits a non trivial solution.

Proof. If $x_i = X_i(z_1, \ldots, z_1^{(r_1)}, \ldots)$, $x_i' = \partial X_i / \partial z_1^{(r_1)} z_1^{(r_1+1)} + \cdots$
and $(\partial X_i / \partial z_1^{(r_1)}, \ldots, \partial X_n / \partial z_1^{(r_1)})$ is a solution of this system.

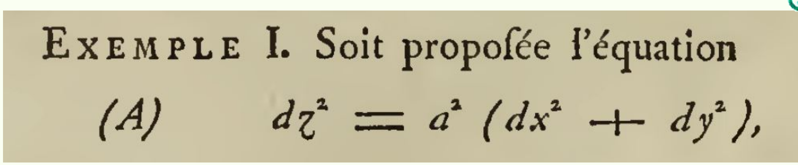
A system that satisfies $x_1' - (x_2')^2 = 0$ is not flat, as the ideal contains $C_1 - 2x_2 C_2 = 0$ and $C_2^2 = 0$, so that $C_1 = C_2 = 0$.

Some History. Monge problem

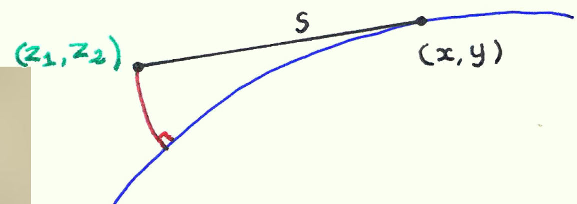
Monge (1784) considered parametrizations such that the independent variable could be parametrized too. Monge problem (Hilbert 1912, Cartan 1914, 1915) is to decide if such a parametrization exists.

Closer to *orbital flatness*, allowing "time control".



EXEMPLE I. Soit propoſée l'équation

(A) $dz^2 = a^2 (dx^2 + dy^2)$,



Monge 1784; $\dot{s}^2 = \dot{x}^2 + \dot{y}^2$ Petitot 1995.

In red, involute of the trajectory in blue.

JACOBI'S BOUND

Carl Gustav Jacob (Jacques) Jacobi (1804-1851)

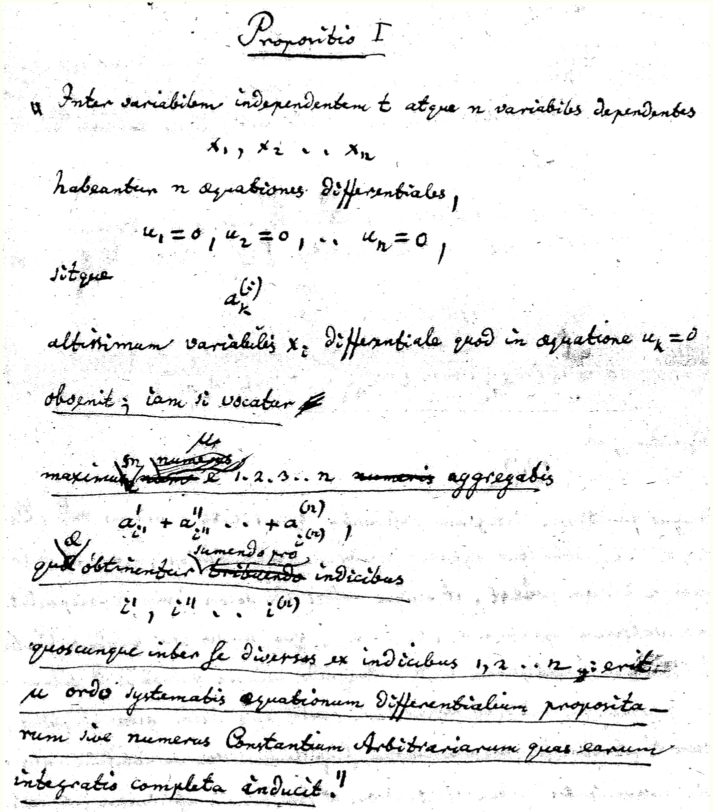
1812-1822 Edict of emancipation
Brother of Moritz Hermann (Moses) Jacobi (1801-1874)
Professor at Königsberg 1827-1843
1841 Bankrupt
Travels in Italy 1843–1844
1844– Berlin, Preußischen Akademie der Wissenschaften
1848 Revolution

Differential equations
Algebraic equations
Number theory
Tropical determinant, Graph theory and shortest paths (1836-1845)

From Königsberg to Berlin via Roma

Posthumous manuscripts…

Most manuscripts are in the Archives of Berlin Academy. Some letters are in Institute Mittag-Leffler. Translations FO 2003–2009.


Propositio I

"Inter variabilem independentem t atque n variabiles dependentes

$x_1, x_2 \,.\,.\, x_n$

habeantur n aequationes differentiales,

$u_1=0,\ u_2=0,\ .\,.\ u_n=0,$

sitque

$a_k^{(i)}$

altissimum variabilis x_i differentiale quod in aequatione $u_k=0$ obvenit; iam si vocatur

μ

maximum e $1.2.3\,.\,.\,n$ aggregatis

$a'_{i'} + a''_{i''} \,.\,.\, + a^{(n)}_{i^{(n)}},$

quae obtinentur sumendo pro indicibus

$i',\ i''\ .\,.\ i^{(n)}$

quoscunque inter se diversos ex indicibus $1, 2 \,.\,.\, n$; erit μ ordo systematis aequationum differentialium propositarum sive numerus Constantium Arbitrariarum quas earum integratio completa inducit."

Proposition I.

"Let

$$u_1 = 0,\ u_2 = 0,\ \dots,\ u_n = 0,$$

be n differential equations between the independent variable t and the dependent variables $x_1, x_2, \dots, x_n$ and let

$$a_k^{(i)}$$

be the maximal order of the variable x_i in the equation $u_k = 0$. *Then, if one calls*

$$\mu$$

the maximum of the $1 \cdot 2 \cdot 3 \cdots n$ *sums*

$$a'_{(i')} + a''_{(i'')} + \cdots + a^{(n)}_{i^{(n)}},$$

that one gets for all different indices $i', i'', \dots, i^{(n)}$ *chosen between indices* $1, 2, \dots, n$; μ *will be the order of the system of differential equations, or also the the number of arbitrary constants that its complete integration makes appear."*

Quasi-regular components

Definition 1. — *Let* $P_i \in \mathcal{F}\{x_1, \dots, x_n\}$, *for* $1 \le i \le n$ *be differential polynomials,* $\mathcal{P}_k$ *be a prime component of* $\{P\} = \bigcup_{k=1}^{q} \mathcal{P}_k$, *and* $\mathcal{G}/\mathcal{F}$ *the differential field extension it defines. It is said to be* quasi-regular *if the family* $\mathrm{d}P_i^{(k)}$, *for* $(i,k) \in [1,n] \times \mathbf{N}$ *is linearly independent in* $\Omega_{\mathcal{G}/\mathcal{F}}$.
Johnson regularity hypothesis. 1969, 1977.

Definition 2. — *Let* $P_i(x_1, \dots, x_n)$, $1 \le i \le n$ *be a differential system. Let* $a_{i,j} := \mathrm{ord}_{x_j} P_i$, *with* $a_{i,j} := -\infty$ *if* x_j *and its derivatives do not appear in* P_i. *We denote by* $A_P := (a_{i,j})$ *the* order matrix.
We call Jacobi number *of the system* $\mathcal{O}_P := \max_{\sigma \in S_n} \sum_{i=1}^{n} a_{i,\sigma(i)}$.

Let $\mathcal{P}$ *be a component of* $\{P\}$. *The order with respect to* $\mathcal{P}$ *is defined by* $\mathrm{ord}_{\mathcal{P},x_j} P_i = \max\{k \in \mathbf{N} | \partial P_i / \partial x_j^{(k)} \notin \mathcal{P}\}$. *We define* $A_{\mathcal{P},P} = (a_{\mathcal{P},i,j})$ *and* $\mathcal{O}_{\mathcal{P},P}$ *accordingly.*

$\mathcal{O}_P$ is the *tropical determinant* of the order matrix A_P.

An improved statement

Definition 3. — *Let* $\nabla_{\mathscr{P},P}$ *be obtained by keeping in the determinant* $|\partial P_i/\partial x_j(a_{\mathscr{P},i,j})|$ *only the terms* $e(\sigma)\prod_{i=1}^n \partial P_i/\partial x_{\sigma(i)}(a_{\mathscr{P},i,\sigma(i)})$ *such that* $\sum_{i=1}^n a_{\mathscr{P},i,\sigma(i)} = \mathcal{O}_{\mathscr{P},P}$. *We call it the* truncated determinant *of* P *with respect to* $\mathscr{P}$

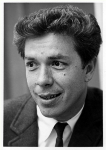
Theorem 4. — *If* $\mathscr{P}$ *is a quasi-regular component of* P, *the order of* $\mathscr{P}$ *is at most* $\mathcal{O}_{\mathscr{P},P}$ *and equal to* $\mathcal{O}_{\mathscr{P},P}$ *iff* $\nabla_{\mathscr{P},P} \notin \mathscr{P}$.

Kondratieva *et al.* 1982 for $\mathcal{O}_P$ without the ∇_P condition. Generalization to PDE 2009.
Sadik and FO 2007 (for diffieties). FO 2022 for diff. algebra.

Canons and minimal covers
Jacobi's algorithm and the Hungarian Method

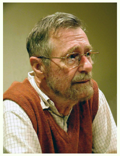
DEFINITION 5. — *A* canon *of A is a vector ℓ such that $(a_{i,j} + \ell_i)$ there exist a permutation $\sigma \in S_n$ with $a_{i,\sigma(i)} + \ell_i$ maximal in its column.*
A cover *of A is defined by vectors μ and ν such that $a_{i,j} \leq \mu_i + \nu_j$.*
To any canon ℓ, we associate a minimal cover *(with $\sum_{i=1}^{n}(\mu_i + \nu_i)$ minimal) defined by $\mu_i = \max_{k=1}^{n} \ell_k - \mu_i$ and $\nu_j = \max_{i=1}^{n} a_{i,j} - \mu_i$.*

Jacobi's algorithm computes "in polynomial time" a unique minimal canon. It is basically equivalent to Kuhn's Hungarian method (1955) for solving the affectation problem, using covers introduced by Jenő Egerváry. Mariage problem with weights.
Mariage problem. Denés Kőnig 1931.

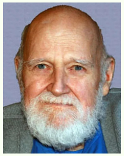
  

Shortest path algorithm

Jacobi gave two other algorithms to compute a minimal canon.



The first, when a canon is known. It is equivalent to Dijkstra algorithm for shortest paths 1959.



The second, when a permutation σ that provides the maximal sum $\sum_{i=1}^{n} a_{i,\sigma(i)}$ is known. It is equivalent to Bellman (1956) and Ford's (1958) algorithm 1956.

Proof of Jacobi's bound

— First idea. Reduce to the linear case. This is quasi-regularity.

— Second idea. Reduce to constant coefficients. If $\nabla_{\mathscr{P},P} \notin \mathscr{P}$, then the order of the module $(\mathrm{d}P)$ considered as a constant coefficient linear system or a time-varying one are the same.
$(\mathrm{d}/\mathrm{d}t)cx_i^{(a)} + \cdots = cx_i^{(a+1)} + \cdots$

– Third idea. $\nabla_{\mathscr{P},P}$ is the coefficient of the term of degree $\mathscr{O}_{\mathscr{P},P}$ of the characteristic polynomial of the square matrix of differential operators associated to $\mathrm{d}P$.

Shortest reduction

Theorem 6. — *Let P_i, $1 \leq i \leq n$ be a system in n differential indeterminates $x_1, \ldots, x_n$ that defines a diffiety V in a neighborhood of a point $\eta \in \mathbf{J}(\mathbf{R}, \mathbf{R}^m)$.*
If ∇_P does not vanish at η, there exists $\sigma \in S_n$ and an open set $W \ni \eta$ such that the diffiety admits in W a normal form

$$x_j^{(\alpha_{\sigma^{-1}(j)}+\beta_j)} = f_j(x),$$

so that the order of the diffiety is $\mathcal{O}_\Sigma$.
Sadik and FO 2007. Differential algebra version FO 2022.

Posterity

Nanson 1876 and Jordan 1883. Heuristic approach for computing a resolvent under foggy genericity hypotheses.

Chrystal 1895. Linear case with constant coefficients.

Ritt 1935, 1950 General linear case and $n \leq 2$.

Volevich 1960. Linear case (differential operators).

Kondratieva, Mikhalev and Pankratiev 1982. (Johnson's regularity hypothesis)

Shaleninov 1990, Pryce 2001. Shortest reduction.

Sadik and FO 2007. Diffieties. Underdetermined systems ∇.

FO 2022. Diff. algebra. Underdetermined systems, ∇, shortest reduction, change of orderings, resolvents.

OUDEPHIPPICAL SYSTEMS

Underdetermined systems

If the number of equation s is smaller than the number of indeterminates, one may consider subsets $Y \subset X$ with $\sharp Y = s$ and Jacobi's number $\mathcal{O}_{Y,P}$ considering only variables in Y.

Then, one may define the the *Jacobi number* of P as $\mathcal{O}_P = \max_{\sharp Y=s} \mathcal{O}_{Y,P}$. It may be computed by completing A_P with $n-s$ rows of 0. It is bounds the order of the system P completed with $n-s$ generic equations of order 0, which generalizes Jacobi's bound to underdetermined systems.

It may be wiser to look for a set Y with $\mathcal{O}_{Y,P}$ minimal.

How to compute the *saddle Jacobi number* of P $\hat{\mathcal{O}}_P := \min_{\sharp Y=s, \mathcal{O}_{Y,P}\neq-\infty} \mathcal{O}_{Y,P}$ efficiently?

By convention if $\mathcal{O}_{Y,\Sigma} = -\infty$ for all Y, or if $s > n$, we set $\hat{\mathcal{O}}_\Sigma = -\infty$. If A is a matrix with entries in $\mathbf{N} \cup \{-\infty\}$, we define $\hat{\mathcal{O}}_A$ accordingly.

Underdetermined systems

Definition 7. — *Systems* Σ *such that* $\hat{\mathcal{O}}_\Sigma = 0$ *are called* oudephippical* systems *or* ō-systems. *A ō-system is called* regular *if there exists* $Y \subset X$ *such that* $\hat{\mathcal{O}}_\Sigma = \mathcal{O}_{Y,\Sigma}$ *and* $\nabla_{Y,\Sigma}$ *does not identically vanish. It is said to be* regular at point η *if there exists* $Y \subset X$ *such that* $\hat{\mathcal{O}}_\Sigma = \mathcal{O}_{Y,\Sigma}$ *and* $\nabla_{Y,\Sigma}$ *does not vanish at* η.

∗ From the Greek: *ouden*, “nothing” and *ephippios*, “saddle”.

Lazy flat parametrization

DEFINITION 8. — *We say that a system* $P \subset \mathcal{O}(\mathbf{J}(\mathbf{R}, \mathbf{R}^n))$ *of* s *differential equations in* n *variables* $x_1, \ldots, x_n$ *admits a lazy flat parametrization at* $\eta \in \mathbf{J}(\mathbf{R}, \mathbf{R}^n)$ *with flat output* Z *if there exists a partition* $X = \{x_1, \ldots, x_n\} = \bigcup_{h=0}^{r} \Xi_h$, *with* $\Xi_0 = Z$, *and an open neighborhood* V *of* η, *such that for all* $0 < h \leq r$ *and all* $x_{i_0} \in \Xi_h$, *there exists an equation* $x_{i_0} - H_{i_0}(\Xi_0, \ldots, \Xi_{r-1})$, *where* H_{i_0} *is a differential function defined on* V *that belongs to the "algebraic" (not differential) ideal generated by* P *in* $\mathcal{O}(V)$.

It is easilly checked that a system P admitting a lazy flat parametrization with flat output $Z = \Xi_0$ is flat.

Main result

THEOREM 9. — *i) A ō-system P, which is regular at point η, admits a* lazy flat parametrization *at point η.*
ii) A system P that admits a lazy flat parametrization *at point η with flat output Z and such that $\nabla_{X\setminus Z,\Sigma}(\eta) \neq 0$ is a regular ō-system at point η.*
iii) If the system P is a ō-system, regular at point η with $\nabla_{Y,P}(\eta) \neq 0$ it is flat at η with flat output $X \setminus Y$.

Efficient criteria

There exists an algorithm that test if $\hat{\mathscr{O}}_P = 0$ in $O(s^{5/2}n)$ operations.

If η is a point of the diffiety defined by P in some open space, then there exists an algorithm to test if P is ō-regular at η in $O(s^4 n)$ elementary operations.

Examples

Goursat normal forms for driftless systems with two imputs.

$$\begin{aligned} z_0' &= v_0; \\ z_i' &= z_{i+1}v_0 \text{ for } 1 \leq i \leq n-2; \\ z_{n-1}' &= v_1, \end{aligned} \tag{1}$$

Affine generalizations Silveira 2010, Silveira, Pereira da Silva, Rouchon 2015.

$$\begin{aligned} z_0' &= v_0; \\ z_i' &= f_i(z_0, z_1, \ldots, z_{i+1}) + z_{i+1}v_0 \text{ for } 1 \leq i \leq n-2; \\ z_{n-1}' &= v_1. \end{aligned} \tag{2}$$

m-chained form. Li, Nicolau, Respondek 2016.

$$
\begin{aligned}
z_0' &= v_0; \\
z_{i,\ell}' &= f_{i,\ell}(z_0, \bar{z}_\ell) + z_{i,\ell+1,} v_0 \text{ for } 1 \leq i \leq m \text{ and } 1 \leq \ell < k; \\
z_{i,k}' &= v_i,
\end{aligned}
\tag{3}
$$

Block diagonal systems

Definition 10. — *An* order 1 block triangular systems *is a system Σ in the variables Ξ such that there exist partitions $\Sigma = \bigcup_{h=1}^{p} \Sigma_h$ and $\Xi = \bigcup_{h=0}^{p} \Xi_h$ such that all equations in Σ_i depend only in variables in $\bigcup_{k=0}^{i} \Xi_k$ and are of order 1 in variables of Ξ_{i-1} and 0 at most in the other variables, with*

i) $\sharp\Xi_{h-1} = \sharp\Sigma_h$*;*

ii) $\mathcal{O}_{X_{i_{h-1}},\Sigma_h} = \sharp\Xi_{h-1}$*;*

iii) $\mathcal{O}_{X_{i_h},\Sigma_h} = 0$*.*

An order 1 block triangular system is said to be chained *at level $h > 0$ if all equations in Σ_h depend only in variables in $\Xi_h \cap \Xi_{h-1}$. It is said to be* stricly chained *if is chained at level h and the equation in Σ_h depend only in derivatives of order 1 of the variables in Ξ_{h-1} and not of those variables themselves.*

Sufficient condition of flat singularity

Theorem 11. — *Let* $\Sigma = \bigcup_{i=1}^{p} \Sigma_h$ *be a block diagonal system in the variables* $\bigcup_{h=0}^{p} \Xi_h$*, chained at level* h_0 *and strictly chained at level* $h_0 - 1$*, and such that all variables in* $\Xi_{h_0-1} \cup \Xi_{h_0}$ *are constants along a given trajectory. Let* η *denote a point of this trajectory.*
Assume moreover that the Jacobian matrix

$$\left(\frac{\partial P}{\partial x} \mid (P, x) \in \Sigma_{h_0} \times \Xi_{h_0}\right) \tag{4}$$

has rank $m_0 < \sharp\Xi_{h_0-1}$ *and that the Jacobian determinants*

$$\left|\frac{\partial P}{\partial x} \mid (P, x) \in \Sigma_{h_0} \times \Xi_{h_0-1}\right| \tag{5}$$

do not vanish at η *for all* $1 \le h \le p$*.*
With these hypotheses, Σ *is not flat at* η*.*

Aircraft equations

12 state variables, 12 equations, 4 controls.

50 parameters.

$\Xi_1 := \{x, y, z\}$: center of gravity.
$\Xi_2 := \{V, \gamma, \chi\}$: speed, flight path angle, aerodynamic azimuth or heading angle.
$\Xi_3 := \{\alpha, \beta, \mu, F\}$: angle of attack, sideslip angle, bank angle, thrust.
$\Xi_4 := \{p, q, r\}$: coordinates of the angular velocity vector.
$\Xi_5 := \{\delta_l, \delta_m, \delta_n\}$: aileron, elevator and rudder deflexion.

Controls: F, δ_l, δ_m, δ_n (may be replaced by $\eta := \frac{F_1-F_2}{F_1+F_2}$ if rudder is lost (differential thrust control).

Aerodynamics. The GNA models

Thanks to the NASA and US tax-payers.

Old planes: Twin Otter, F-4, F-16.

And the GTM, a 5.5% model of a transport plane.

Shape of equations. A nearly chained system

We have:

$\Xi_1' = f_1(\Xi_2)$: Ξ_2 may be computed if $\gamma \neq 0$.
$\Xi_3' = f_3([z], \Xi_2, \Xi_3, \Xi_4)$: Ξ_4 may be computed if $\gamma \neq 0$ (linear).
$\Xi_4' = f_3([z], \Xi_2, \Xi_3, \Xi_4, \Xi_5)$: Ξ_4 may be computed (linear using GNA).

$\Xi_2' = f_1([z], \Xi_2, \Xi_3, [\Xi_4, \Xi_5])$: one may compute Ξ_3 with some simplification.

The simplified model is flat, using flat outputs x, y, z and $\zeta \in \{\alpha, \beta, \mu, F\}$.

Martin 1992 for $\zeta = \beta$

Stalling condition

Maximal lift $\partial\hat{Z}/\partial\alpha = 0$; speed V such that the lift is equal to gravity.

THEOREM 12. — *Let a trajectory be such that α, β, μ, F, γ, χ and V are constants, with moreover $\beta = \mu = 0$, α and V respectively equal to the stall a.o.a. and stall speed. A point eta of this trajectory is not flat.*

Computing the parametrization using power series

Easy computation (lazy flat parametrization).

The evaluation is done with constant step: 1 sec is fine in most situations.

We want to have an evaluation of the δ_i at order 2.

We compute the series for x, y, z at order 6, ζ at order 4.

For V, γ, ξ, we use the formulas: $V = \sqrt{(x')^2 + (y')^2 + (z')^2}$, $\xi = \mathrm{atan}(y'/x')$, ...

For α, β, μ, F, we need to solve a non linear system. Sustitutions reduce to a system of 2 equations, then we use a numerical Newton method for the constant term, and then Newton method for series.

The case of Ξ_4 and Ξ_5 is easy, as the systems are linear.

Data are kept in an array to be used to compute values between t_i and t_{i+1} by interpolation.

Feed-back

We use feed-back to correct model errors.

We work in the linearized system around the planed trajectory.

The differentials δx, δy, ...stand for the differences between the planned values and the values computed by the integrator.

We need to introduce $ix = \int_{t_0}^{t} \delta x(\tau)\mathrm{d}\tau$, ... for better accuracy.

In fact, we take $i\tilde{x} := \int_{t_0}^{t}(\cos(\xi)\delta x + \sin(\xi)\delta y)(\tau)\mathrm{d}\tau$ and $i\tilde{y} := \int_{t_0}^{t}(-\sin(\xi)\delta x + \cos(\xi)\delta y)(\tau)\mathrm{d}\tau$ to preserve symetries.

We compute $\mathrm{d}F$, and the $\mathrm{d}\delta_i$, so that

$$\begin{aligned}
\prod_{i=1}^{3}(\mathrm{d}/\mathrm{d}t - \lambda_{1,i})i\tilde{x} = 0;\\
\prod_{i=1}^{5}(\mathrm{d}/\mathrm{d}t - \lambda_{1,i})i\tilde{y} = 0;\\
\prod_{i=1}^{5}(\mathrm{d}/\mathrm{d}t - \lambda_{1,i})i\tilde{z} = 0;\\
\prod_{i=1}^{3}(\mathrm{d}/\mathrm{d}t - \lambda_{1,i})i\tilde{\zeta} = 0,
\end{aligned}$$

for $\zeta = \beta$ or μ.

Again, the formulas are kept in an array to be used by numerical functions during integration.

The choice of suitable $\lambda_{i,j}$ may be difficult and require many repeated experiments.

Generalized flatness

Basic idea. Step zero. Motion planning using the simplified model, *i.e.* setting p, q, r and the δ_i to 0. It provides an evaluation for p, q, r and the δ_i.

Step $j + 1$. Motion planning, using the values p, q, r and the δ_i computed at step j.

To get the controls δ_i at order 2, we need to start with x, y, z at order $6 + 2J$, going to step J.

Folkloric idea. All systems would be flat if one could use an infinite number of derivatives. *Cf.* Sluis and Rouchon.

Some numerical experiments

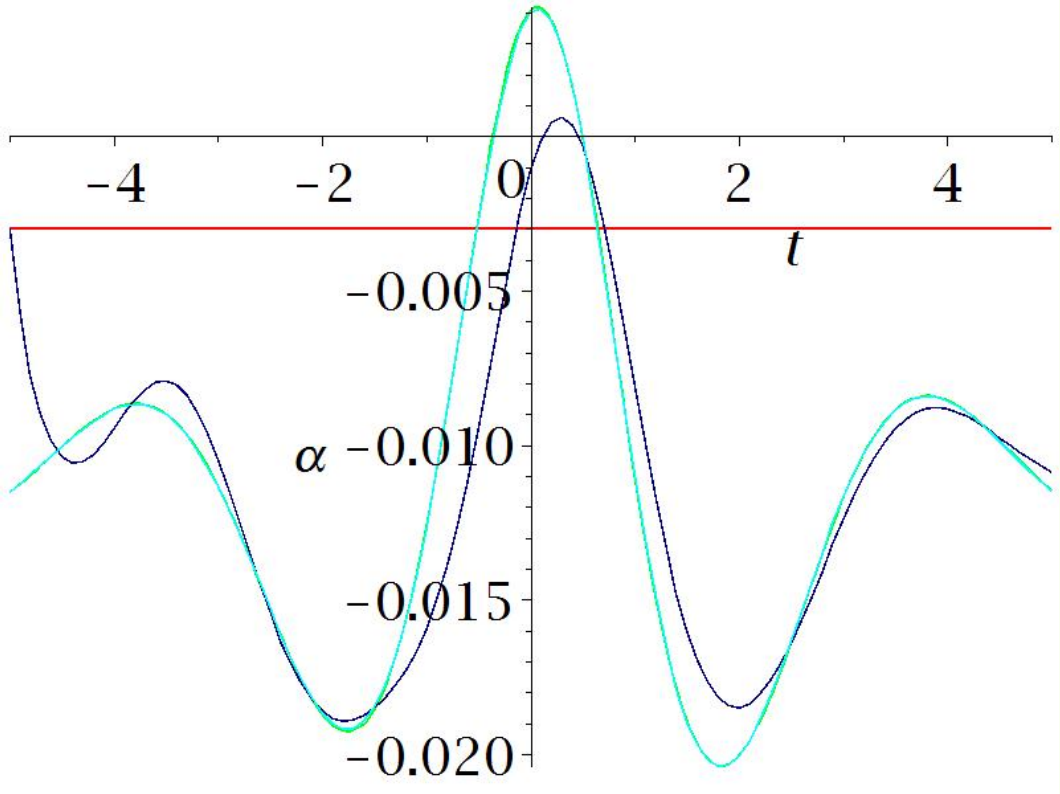
Implementation in Maple.

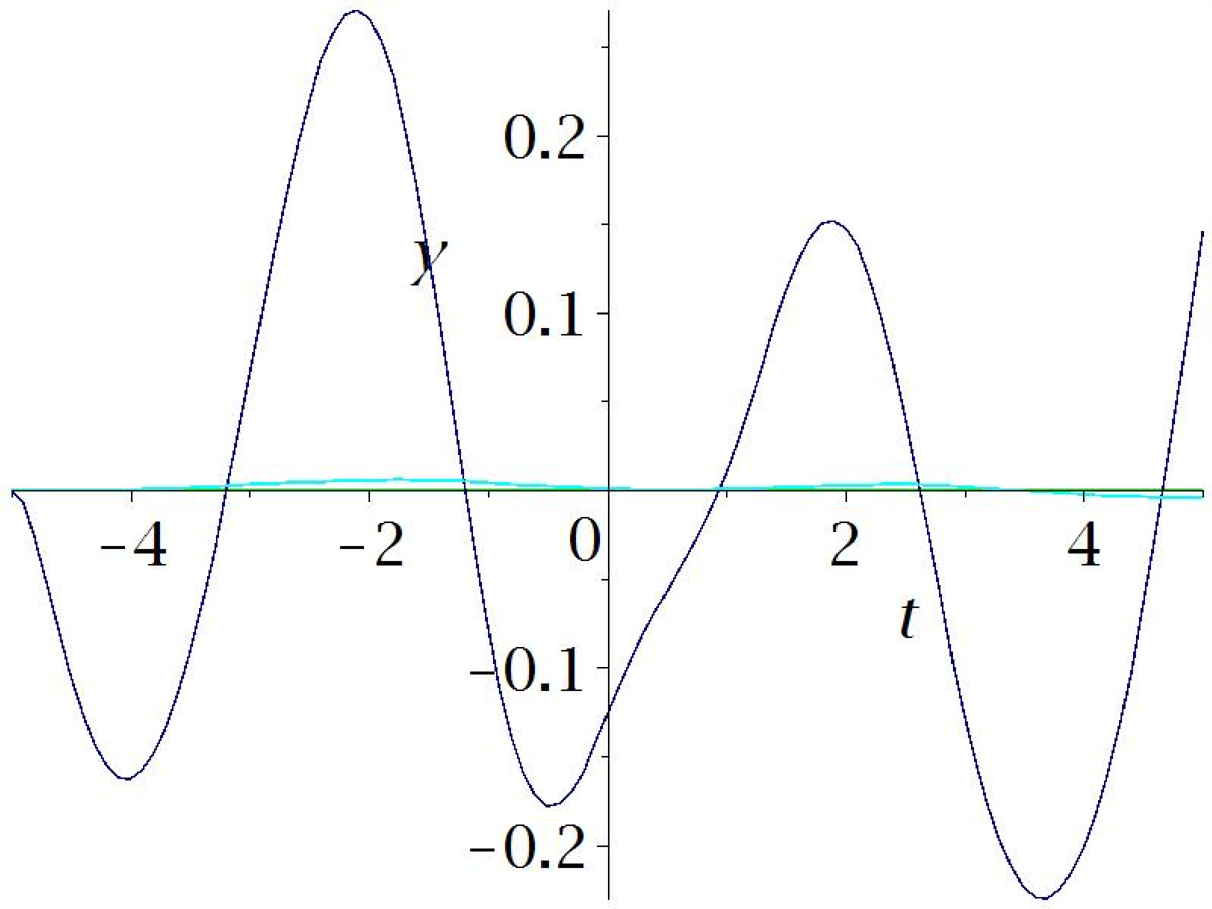
Aileron roll and parabolic flight. GTM

$\lambda_{i,j}$: [2., 2., 2.], [2., 2., 2., 2., 2.], [2., 2., 2., 2., 2.], [2., 2., 2.]

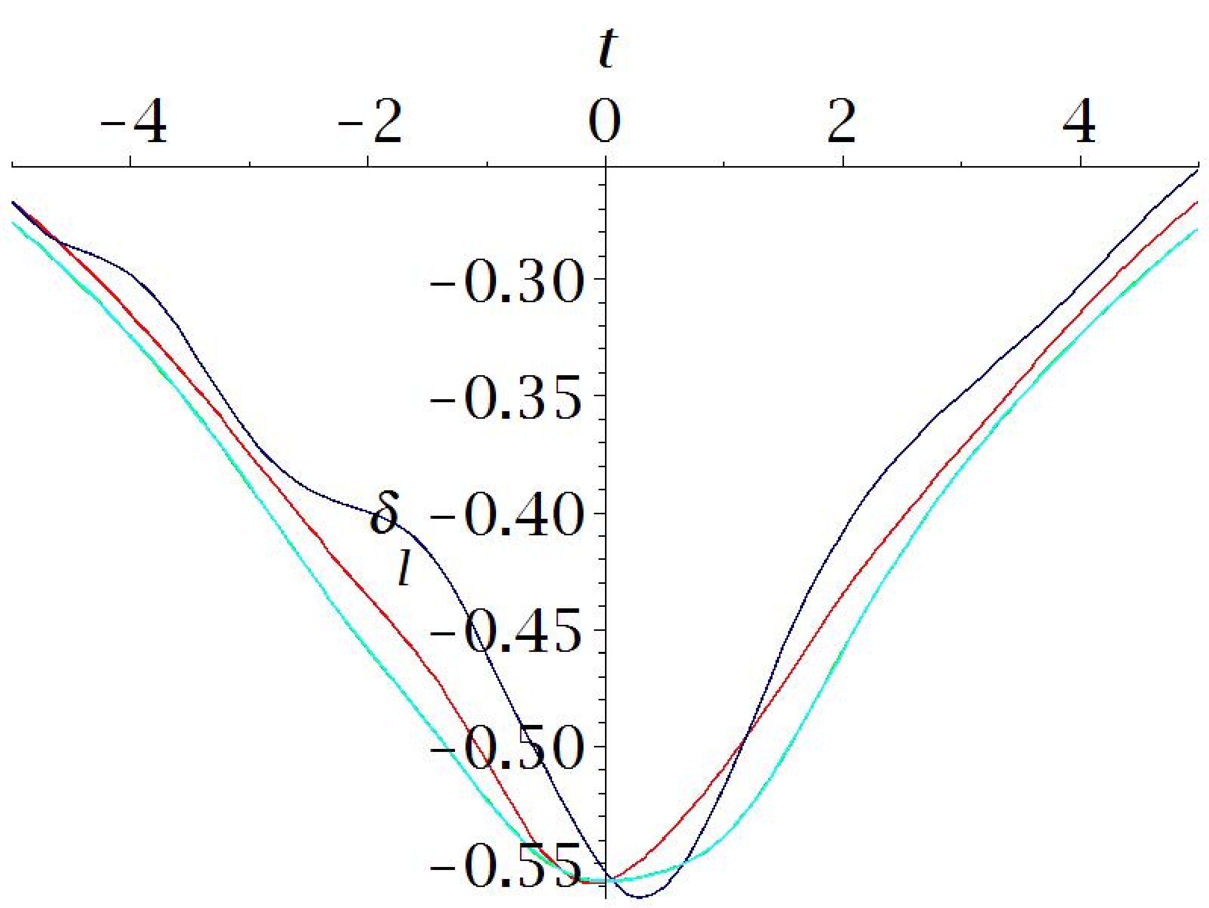
$x = 100\text{km/h}t$, $y = 0.$, $z = -1000 + \text{g}t^2/2$, $\mu = \pi t/2$.

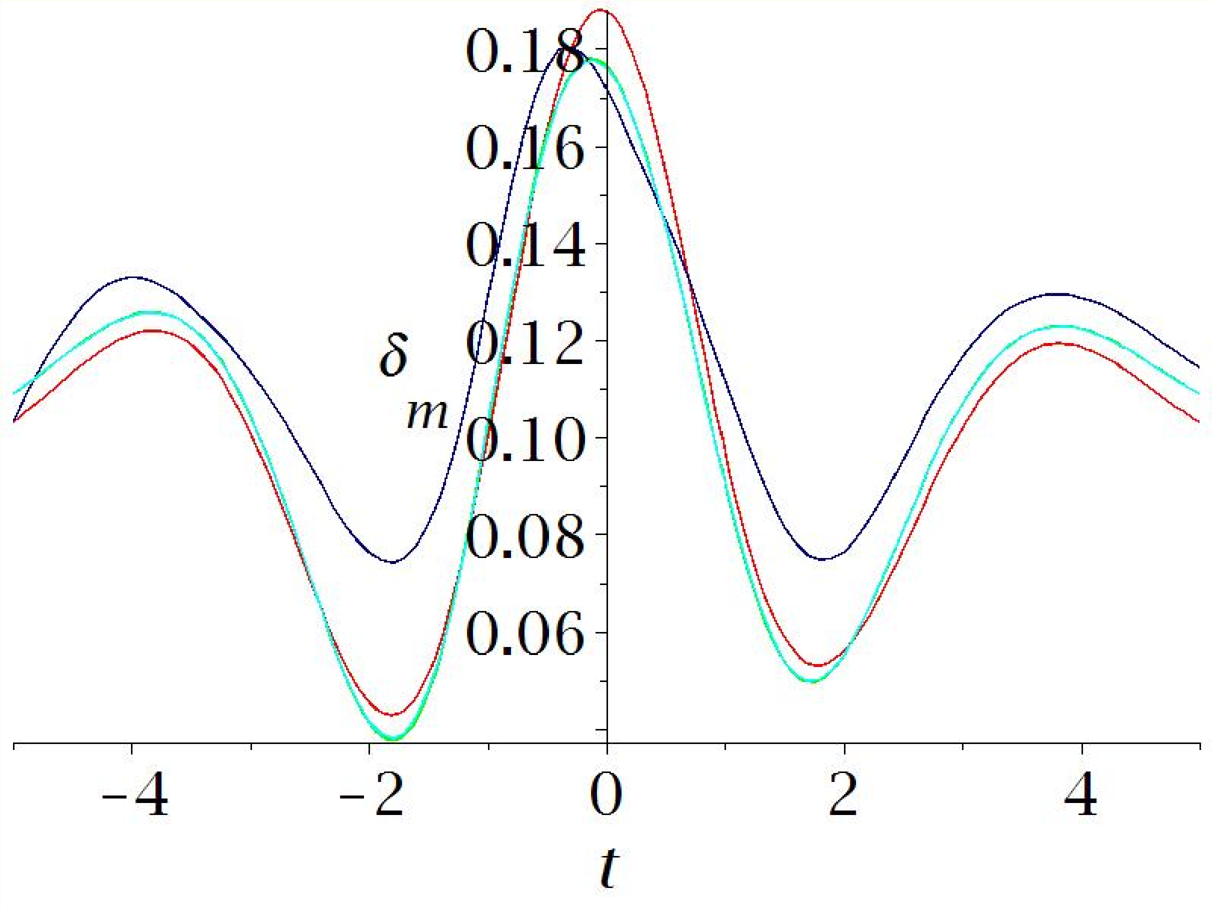
Curves





0.2
0.1
y
-0.1
-0.2
-4
-2
0
2
4
t


t
-4
-2
0
2
4
-0.30
-0.35
δ
l
-0.40
-0.45
-0.50
-0.55


0.18
0.16
0.14
0.12
0.10
0.08
0.06
δ
m
−4
−2
0
2
4
t

Convergence?

Values for the controls F, δ_l, δ_m and δ_n at $t = -1.9$.

	$J = 0$	$J = 1$	$J = 2$	$J = 3$	$J = 4$	$J = 5$	$J = 6$	$J = 7$
F	-2.36	8.40	8.56	8.610	8.624	8.628	8.6304	8.6309
δ_l	-0.44	-0.45	-0.462	-0.4642	-0.4647	-0.4648	-0.46493	-0.464918
δ_m	0.04	0.04	0.039	0.0389	0.0387	0.03872	0.038730	0.038731
δ_n	0.05	0.07	0.085	0.0871	0.087	0.08800	0.087997	0.0880978

References

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Thanks !