On the density of strongly minimal algebraic vector fields

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DART XI, June 2023

Introduction

The subject of my talk concerns the notion of strongly minimal algebraic differential equation which is a property of non linear algebraic differential equations

$$(E) : P(y, y', \ldots, y^{(r)}, t) = 0.$$

This is a notion coming from model theory (in the sense of mathematical logic). Together with definable Galois theory (in the sense of Pillay), this notion lies at the heart of several recent application of model theory to differential algebra (in characteristic 0).

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Plan of my talk :

(1) Present a (nonstandard formulation of) this notion using a weakening of the properties

(C_n) : any n distinct non algebraic solutions of (E) are together with their derivatives of order $< r$ algebraically independent over $\mathbb{C}(t)$.

described in James Freitag's talk.

(2) Describe how to produce "many" examples of autonomous algebraic differential equations satisfying this property.

First, I will start by explaining what I mean by an algebraic differential equation in a geometric set up...

Convention and notation

Throughout my talk, the time variable t will always be a complex variable allowed to move in the complement S of a finite number of points of the complex plane $\mathbb{C}$.

We encode an ODE $P(y, y', \ldots, y^{(r)}, t) = 0$ with $P \in \mathbb{C}[S][x_0, \ldots, x_r]$ as follows

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(1) first, the "static situation" is given by the smooth algebraic variety X defined by

$$P(x_{\mathbf{0}}, \ldots, x_n, t) = 0 \text{ and } \frac{\partial P}{\partial x_n}(x_{\mathbf{0}}, \ldots, x_n, t) \neq 0$$

inside $\mathbb{C}^{n+1} \times S$ with coordinates $(x_0, \ldots, x_n, t)$.

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(3) the "dynamical part" is encoded as an algebraic vector field v on X : the section $v : X \to TX$ of the tangent bundle of X given by

$$v = \frac{\partial}{\partial t} + x_1 \frac{\partial}{\partial x_0} + \ldots + x_n \frac{\partial}{\partial x_{n-1}} + H \frac{\partial}{\partial x_n} \text{ where } H = -\frac{\frac{\partial P}{\partial t} + \sum_{i=1}^{n-1} x_i \frac{\partial P}{\partial x_{i-1}}}{\frac{\partial P}{\partial x_n}}.$$

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Decoding property : the analytic integral curves of the vector field are the curves of the form

$$t \mapsto (y(t), \dots, y^{(r)}(t), t) \text{ where } y(t) \text{ is a regular solution of the ODE.}$$

Algebraic differential equations

Definition

An algebraic differential equation (E) is defined by the following data :

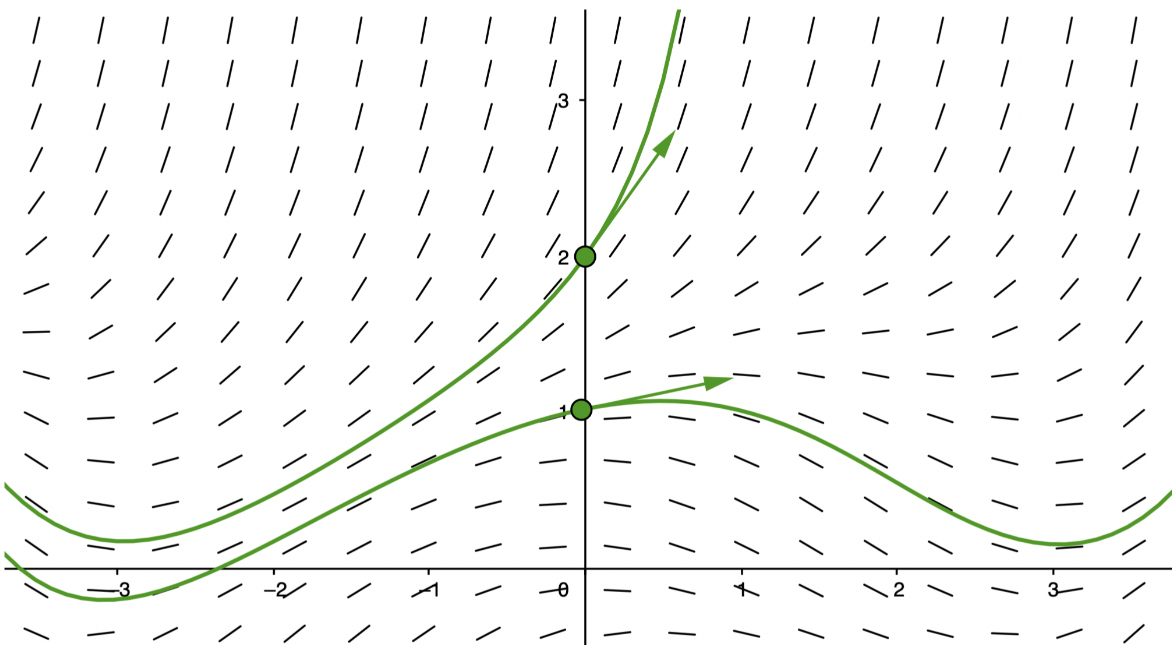
(a) a smooth complex algebraic variety X equipped with the structure of a smooth fibration over S :

$$\text{a surjective submersion } \pi : X \to S$$

whose (complex) fibers are connected (and smooth) complex algebraic varieties.

(b) a vector field $v : X \to TX$ on X extending the vector field $\frac{d}{dt}$ on S induced by t :

$$\text{the differential } d\pi : TX \to TS \text{ satisfies } d\pi_x(v(x)) = \frac{d}{dt}_{|t=\pi(x)} \text{ for all } x \in X.$$



Some classical terminology

Consider an algebraic ODE (E) defined by

$$\pi : (X, v) \to (S, \frac{d}{dt}).$$

(a) the order of the equation is

$$\mathrm{ord}(E) := \dim(X/S) = \dim(X) - 1$$

(b) an analytic solution σ of the equation is a local analytic section of π

$$\sigma : U \subset S^{an} \to X^{an}$$

such that $d\sigma(\frac{d}{dt}_{|t=t_0}) = v(\sigma(t_0))$ for all $t_0 \in U$.

(c) an invariant subvariety of the equation is a closed subvariety Z of X such that :

$$\text{for every } z \in Z_{reg}, v(z) \in T_zZ \subset T_zX$$

Equivalently, the ideal defining Z is a differential ideal for the derivation defined by v.

Remark : If σ is an analytic solution of the equation then

$$\overline{\sigma} := \overline{\sigma(U)}^{Zar} \text{ inside } X$$

is an invariant subvariety of the equation.

Computing transcendence degrees

If (E) is an algebraic ODE and σ is an analytic solution of (E), we set :

$$\mathrm{td}(\sigma/\mathbb{C}(t)) := \dim(\overline{\sigma}/S) = \dim(\overline{\sigma}) - 1.$$

To make sense of quantities of the form

$$\mathrm{td}(\sigma_1, \dots, \sigma_r/\mathbb{C}(t))$$

where $\sigma_1, \dots, \sigma_r$ are analytic solutions defined on the same open set of (possibly different) algebraic ODEs, we use :

Lemma (Fiber products)

Consider two algebraic ODEs

$$\pi_i : (X_i, v_i) \to (S, \frac{d}{dt}) \text{ for } i = 1, 2.$$

There is a unique structure of algebraic differential equation on

$$X_1 \times_S X_2 := \{(x_1, x_2) \in X_1 \times X_2 \mid \pi_1(x_1) = \pi_2(x_2)\}$$

whose solutions on an analytic open set U of S are of the form

$$\sigma_1 \times \sigma_2 : t \mapsto (\sigma_1(t), \sigma_2(t))$$

where σ_1, σ_2 are analytic solutions on U of the first and the second equation respectively.

Strong minimality

If (E) is an algebraic ODE given by $\pi : (X, v) \to (S, \frac{d}{dt})$ we denote by

$$\pi^{[n]} : (X^{[n]}, v^{[n]}) \to (S, \frac{d}{dt})$$

the n^{th} fiber power of (E) over S.

Definition (Strong minimality, equivalent to the classical definition)

Assume that $dim(X) \geq 2$. We say that

$$\pi : (X, v) \to (S, \frac{d}{dt})$$

is strongly minimal if for every $n \geq 1$, we have

$(\mathcal{D}_n)$: every invariant subvariety Z of $(X^{[n]}, v^{[n]})$ satisfies that :
$\mathrm{ord}(E) = \dim(X) - 1$ divides $\dim(Z/S) = \dim(Z) - 1$.

Remarks :

- every algebraic ODE of order one $(\dim(X) = 2)$ is strongly minimal.
- If $m \leq n$ then

$$(\mathcal{D}_n) \Rightarrow (\mathcal{D}_m)$$

The condition $(\mathcal{D}_1)$

Consider an algebraic ODE of order $r \geq 1$

$$\pi : (X, v) \to (S, \frac{d}{dt})$$

(1) The condition $(\mathcal{D}_1)$ gives only two possibilities for an analytic solution of the equation :

$$(*) : \text{ either } \mathrm{td}(\sigma/\mathbb{C}(t)) = 0 \text{ or } \mathrm{td}(\sigma/\mathbb{C}(t)) = r$$

In the first case, σ is an algebraic solution of the equation : $\overline{\sigma}$ is an algebraic curve of X intersecting finitely many times every fiber of π. In the second case, the solution σ is a generic solution of the equation : $\overline{\sigma}$ is Zariski-dense in X.

(2) For every $n \in \mathbb{N} \cup \{\infty\}$, there is a strongly minimal differential equation with exactly n algebraic solutions (hard case : $n = \infty$).

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(3) $(\mathcal{D}_n)$ implies that for every analytic solutions $\sigma_1, \dots, \sigma_n$ of (E)

$$\mathrm{ord}(E) \text{ divides } \mathrm{td}(\sigma_1, \dots, \sigma_n/\mathbb{C}(t))$$

but this condition is weaker...

This is because not every invariant subvariety Z of X admits a "generic analytic solution". For example, the Poizat differential equation

$$y'' = y'/y \text{ (and } t' = 1)$$

admits $y' = 0$ as an invariant subvariety (of dimension 2) but satisfies $(*)$.

The condition $(\mathcal{D}_2)$ gives three possibilities...

$$(*)_2 : \mathrm{td}(\sigma_1, \sigma_2/\mathbb{C}(t)) = 0,\ \mathrm{td}(\sigma_1, \sigma_2/\mathbb{C}(t)) = r \text{ or } \mathrm{td}(\sigma_1, \sigma_2/\mathbb{C}(t)) = 2r.$$

(1) The blue case allows for the existence of nontrivial birational symmetries τ while (C_2) prevents it :

$$\begin{tikzcd} (X, v) \arrow[r, "\tau"] \arrow[d, "\pi"] & (X, v) \arrow[dl, "\pi"] \\ (S, \frac{d}{dt}) & \end{tikzcd}$$

Given a birational symmetry τ and a generic solution σ_1 of the ODE, $\sigma_2 = \tau(\sigma_1)$ is a generic solution of the ODE such that

$$\mathrm{td}(\sigma_1, \sigma_2/\mathbb{C}(t)) = r.$$

(2) $(*)_2$ means that whenever σ_1, σ_2 are two generic solutions of the equation such that

$$\mathrm{td}(\sigma_1, \sigma_2/\mathbb{C}(t)) < 2r$$

then there exists a generically finite-to-finite self-correspondence Z of X over S such that

$$(\sigma_1(t), \sigma_2(t)) \in Z \text{ for all } t \in U.$$

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$$(*)_2 : \mathrm{td}(\sigma_1, \sigma_2/\mathbb{C}(t)) = 0, \ \mathrm{td}(\sigma_1, \sigma_2/\mathbb{C}(t)) = r \text{ or } \mathrm{td}(\sigma_1, \sigma_2/\mathbb{C}(t)) = 2r.$$

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$$(\sigma_1(t), \sigma_2(t)) \in Z \text{ for all } t \in U.$$

(3) If V is a vector bundle over S of rank $r \geq 2$ and v is a linear ODE on V then

$$\begin{tikzcd} (V, v) \arrow[r] \arrow[d, "\pi"] & (\mathbb{P}(V), \mathbb{P}(v)) \arrow[dl] \\ (S, \frac{d}{dt}) & \end{tikzcd}$$

shows that a linear ODE of order ≥ 2 does not satisfy $(\mathcal{D}_2)$ because of

$$Z = V \times_{\mathbb{P}(V)} V \text{ inside } V \times_S V$$

The conditions $(\mathcal{D}_n)$ for $n \geq 3$

Consider an algebraic ODE (E) given by

$$\pi : (X, v) \to (S, \frac{d}{dt}).$$

Theorem (Theorem A, with J. Freitag and R. Moosa)

$$(\mathcal{D}_3) \Rightarrow (\mathcal{D}_n) \textit{ for all } n \geq 1$$

If V is a vector bundle over S of rank $r \geq 3$ and v is a linear ODE on V with Galois group containing $SL_r(\mathbb{C})$ then the "Riccati equation"

$$\pi : (\mathbb{P}(V), \mathbb{P}(v)) \to (S, \frac{d}{dt})$$

satifies $(\mathcal{D}_2)$ and not $(\mathcal{D}_3)$ (Freitag-Moosa).

Some examples of strongly minimal equations

- The Manin Kernels (Hrushovski, 1996)

$$(M) : \begin{cases} y^2 = x(x-1)(x-t) \\ \frac{y}{2(x-t)^2} + (t(t-1)x'/y')' + t(t-1)\frac{x'y'}{y^2} = 0 \end{cases}$$

(M) has infinitely many algebraic solutions, an infinite group of symmetries and a commutative group law compatible with its ODE structure.

- For suitable choices of parameters, the Painlevé equations (Nagloo-Pillay 2011, Umemuera, Watanabe...). For example,

  $$y'' = 2y^3 + ty + \alpha$$

  is strongly minimal if and only if $\alpha \notin \mathbb{Z} + 1/2$
- For suitable choices of $R(y) \in \mathbb{C}(y)^{alg}$, the Schwarzian equations

  $$(y''/y')' - 1/2(y''/y')^3 + R(y)(y')^2 = 0.$$

  (Blazquez-Sanz, Casale, Freitag, Nagloo 2020).
- A Poizat equation

  $$y''/y' = f(y) \text{ where } f(y) \in \mathbb{C}(y)$$

  satisfies $(\mathcal{D}_3)$ if and only if $f(y)$ admits a non zero residue. (Freitag, J., Marker, Nagloo, 2022)

Few examples but a generic condition...

Question (Poizat, 1980)

Is is true that "most" algebraic differential equations are strongly minimal ?

(1) Using Theorem A, Devilbiss and Freitag (2021+) have shown that Poizat's question has a positive answer if one allows algebraic differential equations

$$P(y, y', \dots, y^{(n)}) = 0$$

to have coefficients which are differentially transcendental meromorphic functions.

(2) Another natural setting to inquire about Poizat's question is the previous setting :

$$\pi : (X, v) \to (S, \frac{d}{dt})$$

where the coefficients of the ODE are rational functions of the time variable t.

The second part of my talk will give a complete answer to Poizat's question in the case where the coefficients of the ODE are complex numbers (independent of the time t).

The autonomous assumption

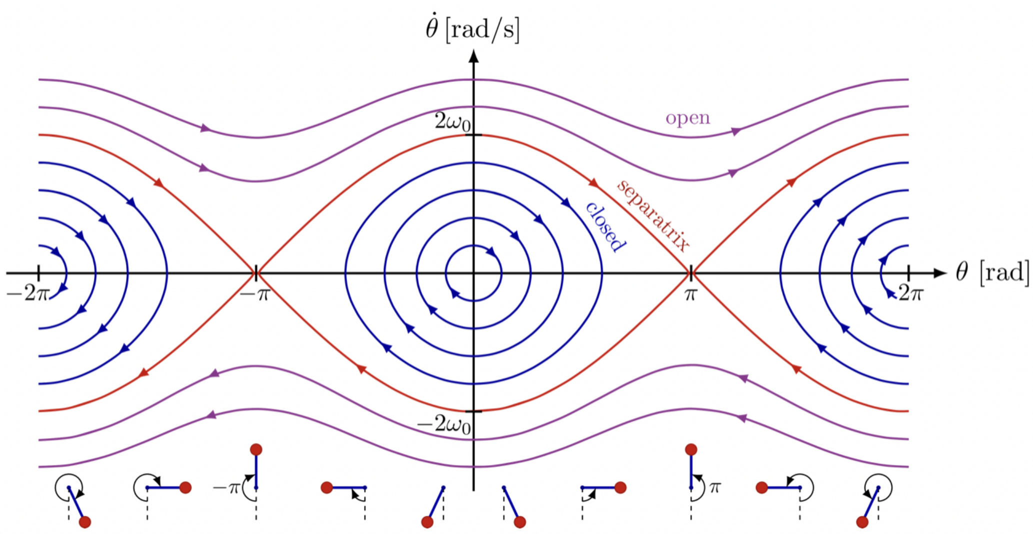
Definition

An autonomous ODE is an algebraic differential equation :

$$\pi : (X, v) \to (S, \frac{d}{dt})$$

such that there exists an algebraic vector field (X_0, v_0) and an isomorphism of algebraic vector fields :

$$\begin{tikzcd} (X, v) \arrow[r, "\cong"] \arrow[d, "\pi"] & (X_0, v_0) \times (S, \frac{d}{dt}) \arrow[dl, "\pi_S"] \\ (S, \frac{d}{dt}) & \end{tikzcd}$$



Consequences of the autonomous assumption

Let (X_0, v_0) be an algebraic vector field and consider the associated autonomous ODE :

$$\pi_S : (X_0, v_0) \times (S, \frac{d}{dt}) \to (S, \frac{d}{dt}).$$

Theorem (Theorem A' with J. Freitag and R. Moosa)

Consider the following two conditions :

- $(\mathcal{D}_1^{aut})$ *: the only proper closed invariant subvarieties of* (X_0, v_0) *are the singular points of the algebraic vector field* (X_0, v_0).
- $(\mathcal{D}_2^{aut})$ *: the proper closed invariant subvarieties of* $(X_0, v_0) \times (X_0, v_0)$ *are all of the form :*
  1. $\{x_1\} \times \{x_2\}$ *where* $x_1, x_2 \in \mathrm{Sing}(X_0, v_0)$,
  2. $\{x_1\} \times X_0$ *and* $X_0 \times \{x_2\}$ *where* $x_1, x_2 \in \mathrm{Sing}(X_0, v_0)$
  3. *generically finite-to-finite correspondences of* X_0 *: closed subvarieties* Z *of* $X_0 \times X_0$ *such that*

$$\text{for } i = 1, 2, \pi_{i|Z} : Z \to X_0 \text{ is dominant and generically finite.}$$

Then

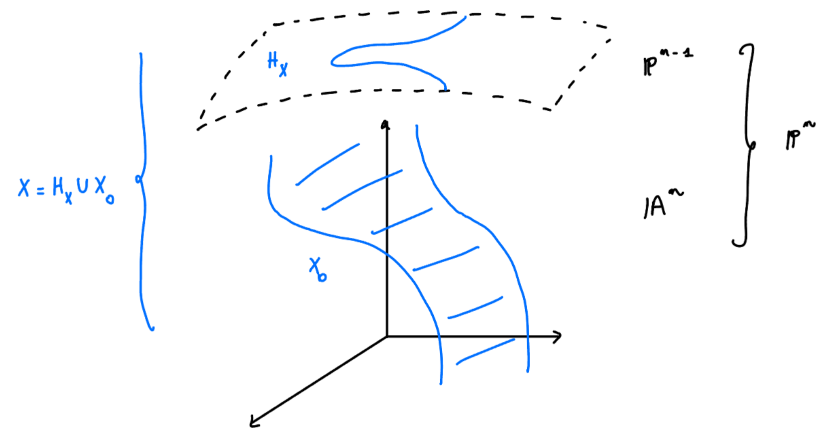
$$(\mathcal{D}_1^{aut}) \text{ and } (\mathcal{D}_2^{aut}) \Leftrightarrow \text{the associated ODE satisfies } (\mathcal{D}_n) \text{ for all } n \geq 1$$

Remark : In that case, a theorem of Hrushovski and Sokolovic (1996) ensures that the equation is always geometrically trivial.

A density theorem

Let X be a smooth projective algebraic variety of dimension $n \geq 2$.

- $X \subset \mathbb{P}^N(\mathbb{C})$ a closed embedding
- H an hyperplane of $\mathbb{P}^N(\mathbb{C})$ such that $H_X = X \cap H$ is smooth.
- Set $X_0 = X \setminus H_X$



We denote by $\Xi(X_0, d)$ the space of vector fields on X_0 with a pole of order $\leq d$ along H_X.

Theorem (Theorem B)

Consider

$$\mathrm{Exp}(X_0, d) = \{v_0 \in \Xi(X_0, d) \mid \textit{ the ODE associated to } (X_0, v_0) \textit{ is not strongly minimal}\}$$

There exists $d_0 = d_0(X, H_X) \geq 0$ such that for all $d \geq d_0$,

$$\mathrm{Exp}(X_0, d) \subset \bigcup_{i \in \mathbb{N}} Z_i \subsetneq \Xi(X_0, d)$$

is contained in a countable union of proper Zariski-closed subsets of $\Xi(X_0, d)$.

In particular, $\mathrm{Exp}(X_0, d)$ has Lebesgue measure zero in $\Xi(X_0, d)$.

Geometric input

Consider $\Xi(X_0, d)$ the space of vector fields on X_0 with a pole of order $\leq d$ along H_X.

$$\mathrm{WExp}(X_0, d) := \{v_0 \in \Xi(X_0, d) \mid (X_0, v_0) \text{ does not satisfies } (\mathcal{D}_1^{aut})\}$$

namely the set of vector fields which admit a proper invariant subvariety which is not a singular point.

Theorem (Coutinho-Pereira, 06')

There exists $d_0 = d_0(X, H_X) \geq 0$ such that for every $d \geq d_0$,

$$\mathrm{WExp}(X_0, d) \subset \bigcup_{i \in \mathbb{N}} Z_i \subsetneq \Xi(X_0, d)$$

is contained in a countable union of proper Zariski-closed subsets of $\Xi(X_0, d)$.

This is not enough as concrete examples show that in general

$$\mathrm{WExp}(X_0, d) \subsetneq \mathrm{Exp}(X_0, d).$$

For instance, the planar vector field

$$\begin{cases} x' = 1 \\ y' = xy + \alpha \end{cases} \quad \text{for } \alpha \neq 0$$

satisfies $(\mathcal{D}_1^{aut})$ but not $(\mathcal{D}_2^{aut})$.

Getting to $n = 1$...

Let (X_0, v_0) be an algebraic vector field and consider

$$\pi : (X_0, v_0) \times (S, \frac{d}{dt}) \to (S, \frac{d}{dt})$$

the associated ODE.

Theorem (Theorem C)

Assume that (X_0, v_0) admits at least one non resonant singular point. Then

$$(\mathcal{D}_1^{aut}) \Leftrightarrow (\mathcal{D}_1^{aut}) + (\mathcal{D}_2^{aut}) \Leftrightarrow \text{ the associated ODE satisfies } (\mathcal{D}_n) \text{ for all } n \geq 1$$

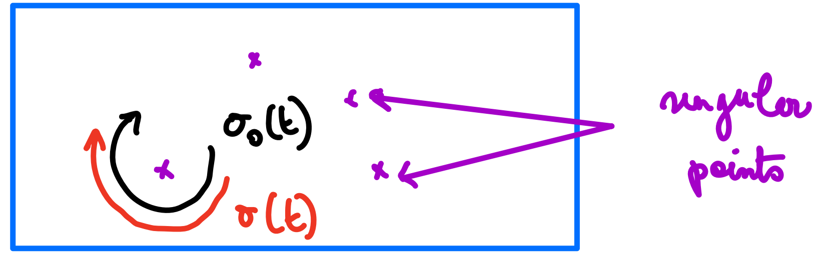
Around any singular point $x_0 \in \mathrm{Sing}(X_0, v_0)$, choosing an analytic (or formal) system of coordinates centered at x_0, we can write

$$v(x_1, \ldots, x_n) = A.(x_1 \frac{\partial}{\partial x_1}, \ldots, x_n \frac{\partial}{\partial x_n}) + \text{h.o.t. in } x_1, \ldots, x_n$$

where $A \in Mat_n(\mathbb{C})$. The singular point is non resonant if the eigenvalues $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ of A are $\mathbb{Z}$-linearly independent.

Strategy of the proof : construction of a linearization

$$
\begin{cases}
y_1' &= f_1(y_1, \ldots, y_n) \\
&\vdots \\
y_n' &= f_n(y_1, \ldots, y_n)
\end{cases}
$$



- Choose $\sigma_0(t) = (y_1(t), \ldots, y_n(t))$ a generic solution defined on U. As diff. fields

$$(\mathbb{C}(x_1, \ldots, x_n), f_1 \frac{\partial}{\partial x_1} + \ldots f_n \frac{\partial}{\partial x_n}) \simeq \mathbb{C}(\sigma_0(t)) = \mathbb{C}(y_1(t), \ldots, y_n(t)) \subset \mathcal{M}(U).$$

- Choose a second generic solution $\sigma(t)$, set $\epsilon(t) = \sigma(t) - \sigma_0(t)$ and compute up to order one in ϵ

$$
\begin{aligned}
\epsilon'(t) &= \sigma'(t) - \sigma_0'(t) = v(\sigma(t)) - v(\sigma_0(t)) \\
&= L_v(\sigma_0(t)) \cdot \epsilon(t) + \text{h.o.t in } \epsilon(t) \text{ where } L_v = \left(\frac{\partial f_i}{\partial x_j}\right)_{i,j} \in Mat_n(\mathbb{C}(x_1, \ldots, x_n))
\end{aligned}
$$

Definition

The linearization of the ODE along a generic solution is the linear differential equation

$$Z' = L_v \cdot Z \text{ over the differential field } (\mathbb{C}(x_1, \ldots, x_n), f_1 \frac{\partial}{\partial x_1} + \ldots f_n \frac{\partial}{\partial x_n})$$

Step 1 : Galois theory of the linearization

$$\begin{cases} y_1' & = f_1(y_1, \dots, y_n) \\ & \vdots \\ y_n' & = f_n(y_1, \dots, y_n) \end{cases}$$

satisfying $(\mathcal{D}_1^{aut})$.

the linearization (L_S)

$$Z' = L_v \cdot Z$$

defined over $(\mathbb{C}(x_1, \dots, x_n), \partial_S)$.

To the linearization (L_S), differential Galois theory associates :

- a connected linear algebraic group (the Galois group of the equation)

$$G = \mathrm{Gal}(L_S/\mathbb{C}(x_1, \dots, x_n)^{alg})$$

- a representation (well-defined up to conjugation)

$$\rho_L : G \to GL_n(\mathbb{C})$$

Proposition (Assuming $(\mathcal{D}_1^{aut})$)

If ρ_L admits a unique proper nontrivial sub-representation then the vector field (X_0, v_0) satisfies $(\mathcal{D}_2^{aut})$.

This follows from an analysis of the property $(\mathcal{D}_2^{aut})$ carried out in a paper with R. Moosa.

Step 2 : Computations of the subrepresentations

Consider the algebraic vector field (X_0, v_0) given by

$$\begin{cases} y_1' & = f_1(y_1, \dots, y_n) \\ & \vdots \\ y_n' & = f_n(y_1, \dots, y_n) \end{cases}$$

with a singularity at 0.

- In Galois theoretic terms, the singularity is non resonant means that the "local" Galois group
$$\mathrm{Gal}(L_S/\mathbb{C}((x_1, \dots, x_n))^{alg}) \simeq \mathbb{G}_m^n.$$

Step 2 : Computations of the subrepresentations

Consider the algebraic vector field (X_0, v_0) given by

$$\begin{cases} y_1' &= f_1(y_1, \ldots, y_n) \\ &\vdots \\ y_n' &= f_n(y_1, \ldots, y_n) \end{cases}$$

with a singularity at 0.

- In Galois theoretic terms, the singularity is non resonant means that the "local" Galois group
$$\mathrm{Gal}(L_S/\mathbb{C}((x_1, \ldots, x_n))^{alg}) \simeq \mathbb{G}_m^n.$$
- The following geometric interpretation explains how to compute the subrepresentation of ρ_L

$$\begin{array}{ccc} \text{subrepresentations of} & \longrightarrow & \text{foliations } \mathcal{F} \text{ on } X_0 \text{ satisfying} \\ \rho_L : Gal(L/\mathbb{C}(x_1, \ldots, x_n)) \to Gl_n(\mathbb{C}) & \longleftarrow & [v_0, \mathcal{F}] \subset \mathcal{F} \end{array}$$

- Using methods of Pereira, the right hand side can be controlled using only one non resonant singularity (a local assumption) and $(\mathcal{D}_1^{aut})$ (a global assumption).

Thank you for your attention!