Hermite Reduction for D-finite Functions via Integral Bases

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Joint work with Shaoshi Chen and Manuel Kauers.

Symbolic integration

Let A be a class of functions. (e.g. $A = \mathbb{Q}(x)$, $\mathbb{Q}(x)(\sqrt{x})$)

Integration Problem. Given $f(x) \in A$, decide whether $\exists\ g(x) \in A$ s.t.

$$f(x) = g'(x).$$

If such a g exists, we say f is integrable in A.

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Additive Decomposition Problem. Given $f \in A$, compute $g, r \in A$ s.t.

$$f = g' + r$$

with the following two properties:

- (minimality) r is minimal in some sense,
- (integrability) f is integrable in $A \quad \Leftrightarrow \quad r = 0$.

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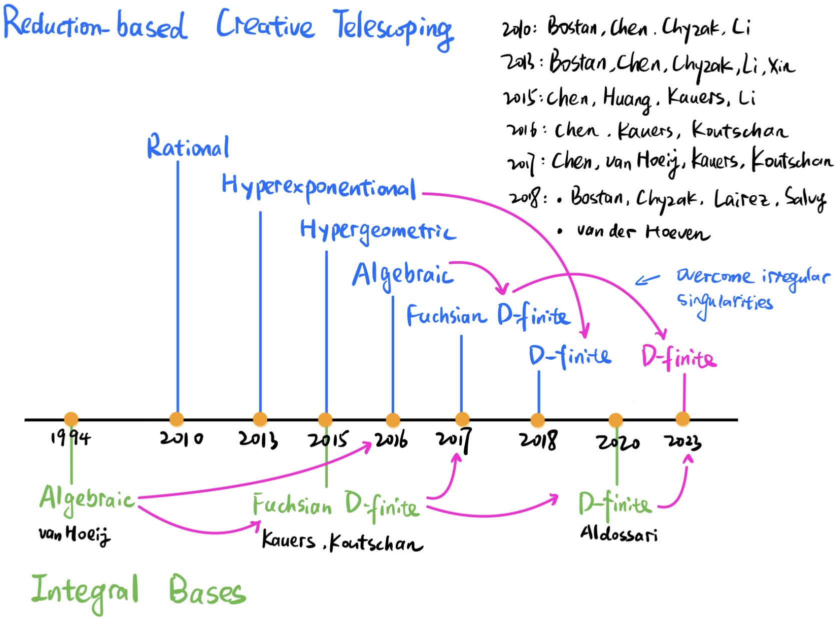
Creative Telescoping Problem (Motivation). If $f \in A$ depends on x and t, find $g \in A$ and a nonzero linear differential operator $L(t, D_t)$ s.t.

$$\underbrace{L(t, D_t)}_{\text{telescoper}}(\ f\ ) = D_x(\underbrace{\ g\ }_{\text{certificate}})$$


Reduction-based Creative Telescoping
2010: Bostan, Chen, Chyzak, Li
2013: Bostan, Chen, Chyzak, Li, Xin
2015: Chen, Huang, Kauers, Li
2016: Chen, Kauers, Koutschan
2017: Chen, van Hoeij, Kauers, Koutschan
2018: • Bostan, Chyzak, Lairez, Salvy
• van der Hoeven
Rational
Hyperexponentional
Hypergeometric
Algebraic
Fuchsian D-finite
D-finite
D-finite
overcome irregular singularities
1994
2010
2013
2015
2016
2017
2018
2020
2023
Algebraic
van Hoeij
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Kauers, Koutschan
D-finite
Aldossari
Integral Bases

Hermite reduction

Rational case.

- Theorem (Ostrogradsky 1845, Hermite 1872). Let $f \in C(x)$. Then

$$f = g' + h \quad \text{with} \quad h = \frac{a}{b},$$

where $\deg_x(a) < \deg_x(b)$ and b is squarefree. Moreover

$$f \text{ is integrable in } C(x) \quad \Leftrightarrow \quad a = 0.$$

Hermite reduction

Algebraic case.

- Theorem (Trager 1984). Let $f \in A = C(x)[y]/\langle m \rangle$ and $\{w_1, \ldots, w_n\}$ be an integral basis. Then

$$f = g' + h \quad \text{with} \quad h = \frac{\sum_{i=1}^{n} a_i w_i}{b},$$

where $a_i, b \in C[x]$ and b is squarefree.

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Remark.

- How to decide the integrability of f in A?

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- How to decide the integrability of f in A? (Trager 1984; Chen, Kauers, Koutschan, 2016.)

Integral bases: two cases

Algebraic case

- $A = C(x)[y]/\langle m\rangle$, where $m \in C(x)[y]$ is irreducible
- $f \in A$ is integral iff for all $\alpha \in \bar{C}$, $f(b)$ is integral for each Puiseux series solution b of m at $x = \alpha$
- The integral elements of A form a free $C[x]$-module.
- Computation: van Hoeij's algorithm 1994, etc.

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D-finite case

- $A = C(x)[D]/\langle L \rangle$, $Dx = xD + 1$, where $L \in C(x)[D]$.
- $f \in A$ is integral iff for all $\alpha \in \bar{C}$, $f(b)$ is integral for each generalized series solution b of L at $x = \alpha$
- The integral elements of A form a free $C[x]$-module.
- Computation: Kauers and Koutschan's algorithm 2015, Aldossari and van Hoeij's algorithm 2020.

D-finite functions

Definition. A function $f(x)$ is called D-finite over $C[x]$ if

$$p_0(x)\, f(x) + p_1(x) f'(x) + \cdots + p_n(x) f^{(n)}(x) = 0 \quad \text{for } p_i \in C[x].$$

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Examples.

$$\frac{1}{x^2+2x}, \quad \frac{1}{\sqrt{x+1}}, \quad \exp(x), \quad \log(x), \quad \cdots$$

Question. Assume y satisfies $x^3 y''(x) + (3x^2+2)y(x) = 0$. Compute

$$\int y\, dx$$

D-finite functions: solution space

Setting.

- $L = p_0(x) + p_1(x)D + \cdots + p_n(x)D^n \in C[x][D]$ with $p_r \neq 0$.
- $A = C(x)[D]/\langle L \rangle$, $Dx = xD + 1$.
- $1 \in A$ represents a solution y of L. Indeed, $L \cdot 1 = L = 0$ in A.

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- Operator action: for any function $f(x)$

$$L \cdot f = p_0(x)f(x) + p_1(x)f'(x) + \cdots + p_n(x)f^{(n)}(x).$$

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- Solution space:

$$L \cdot f = 0$$

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Theorem. Let $L \in C(x)[D]$. Then for each $\alpha \in \bar{C}$, L admits n linearly independent generalized series solutions of the form

$$\exp(P((x-\alpha)^{-\frac{1}{s}})) \quad (x-\alpha)^{\nu} \quad Q((x-\alpha)^{\frac{1}{s}}, \log(x-\alpha))$$

where $s \in \mathbb{N}$, $P \in \bar{C}[x]$, $\nu \in \bar{C}$, $Q \in \bar{C}[[x]][y]$ with $Q(0,y) \neq 0$.

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exponential part

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exponential part — polynomial part — series part + logarithmic part

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- $\exp(P((x-\alpha)^{-\frac{1}{s}}))$: exponential part
- $(x-\alpha)^{\nu}$: polynomial part
- $Q((x-\alpha)^{\frac{1}{s}}, \log(x-\alpha))$: series part + logarithmic part

where $s \in \mathbb{N}$, $P \in \bar{C}[x]$, $\nu \in \bar{C}$, $Q \in \bar{C}[[x]][y]$ with $Q(0,y) \neq 0$.

Definition. Such a series is called integral at α if $\nu \geq 0$.

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Remark.

- If $P = 0$, $s = 1$ for all $\alpha \in \bar{C} \cup \{\infty\}$, then L is called Fuchsian.

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- Hermite reduction for Fuchsian D-finite functions:

Chen, van Hoeij, Kauers, Koutschan 2017

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Example. $f = \exp(x^{-2})$ is integral at 0, but

$$f' = -2x^{-3}\exp(x^{-2}) \text{ is not integral at } 0.$$

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Example. $f = \exp(x^{-2})$ is integral at 0, but

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The valuation ν is decreased by 3, not 1.

This is different from the algebraic and Fuchsian cases.

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Definition. An element $P \in A = C(x)[D]/\langle L \rangle$ is called integral if for each $\alpha \in \bar{C}$

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Theorem. The set $\mathcal{O}$ of all integral elements of A forms a free $C[x]$-module of rank n.

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Definition. A basis for this module $\mathcal{O}$ is called an integral basis.

D-finite functions: obstacle

Consider $A = C(x)[D]/\langle L \rangle$ as a left $C(x)[D]$-module.

Let $W = \{w_1, \ldots, w_n\}$ be an $C(x)$-vector space basis of A. Then

$$W' = \frac{1}{e} M W, \quad \text{where} \quad e \in C[x],\ M \in C[x]^{n \times n}$$

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Problem. If W is an integral basis, then e may not be squarefree for non-Fuchsian D-finite functions.

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Example. Let $L = x^3 D^2 + (3x^2 + 2)D \in \mathbb{Q}(x)[D]$ with two solutions

$$y_1(x) = 1 \quad \text{and} \quad y_2(x) = \exp(x^{-2}).$$

Then $W = \{1, x^3 D\}$ is an integral basis, but $e = x^3$ is not squarefree.

$$\begin{pmatrix} 1 \\ x^3 D \end{pmatrix}' = \frac{1}{x^3} \begin{pmatrix} 0 & 1 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ x^3 D \end{pmatrix}$$

D-finite functions: an example

Example. Let $L = x^3 D^2 + (3x^2+2)D \in \mathbb{Q}(x)[D]$.

- $W = \{\omega_1, \omega_2\} = \{1, x^3 D\}$ is an integral basis.

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- $W = \{\omega_1, \omega_2\} = \{1, x^3 D\}$ is an integral basis.
- Consider
$$f = \frac{a_1 \omega_1 + a_2 \omega_2}{u v^k},$$
where
$$a_1 = -2x^2 - x^4, \quad a_2 = -2 + 3x^2 - 3x^4,$$
$$u = 1, \qquad v = x, \qquad k = 4.$$

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$$u = 1, \qquad v = x, \qquad k = 4.$$

- Task. Reduce the multiplicity k at v.

$\rightsquigarrow$ the goal of Hermite reduction for D-finite functions

D-finite functions: an example

Example. Let $L = x^3 D^2 + (3x^2+2)D \in \mathbb{Q}(x)\langle D\rangle$.

- Hermite reduction. Find $b_1, b_2, c_1, c_2 \in \mathbb{Q}[x]$ such that

$$\frac{a_1\omega_1 + a_2\omega_2}{uv^k} = \left(\frac{b_1\omega_1 + b_2\omega_2}{v^{k-1}}\right)' + \frac{c_1\omega_1 + c_2\omega_2}{uv^{k-1}}.$$

  Task. Reduce the multiplicity k to $k-1$.

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- Multiplying by uv^{k+2} and module v^3 yield that

$$v^2(a_1\omega_1 + a_2\omega_2) \equiv b_1uv^{k+2}\left(\frac{\omega_1}{v^{k-1}}\right)' + b_2uv^{k+2}\left(\frac{\omega_2}{v^{k-1}}\right)' \bmod v^3$$

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- Equating the coefficients of ω_i, we have

$$\begin{pmatrix} a_1v^2 \\ a_2v^2 \end{pmatrix} = \begin{pmatrix} -3x^2 & 0 \\ 1 & -3x^2-2 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \bmod v^3$$

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But the linear system has a solution.

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$$v^2(a_1\omega_1 + a_2\omega_2) \equiv b_1(-3x^2\omega_1 + \omega_2) - (3x^2+2)b_2\omega_2 \mod v^3.$$

- So $b_1 = \frac{2}{3}x^2$, $b_2 = \frac{4}{3}x^2$.

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Example. Let $L = x^3D^2 + (3x^2+2)D \in \mathbb{Q}(x)\langle D\rangle$.

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- Consequently,

$$f = \left(\frac{2x^2\omega_1 + 4x^2\omega_2}{3x^3}\right)' + \frac{(-4-3x^2)\omega_1 + (13-9x^2)\omega_2}{3x^2}$$

D-finite functions: Hermite reduction

Let $W = (\omega_1, \ldots, \omega_n)$ be an integral basis of A and let $eW' = MW$.

Problem. Let v be a squarefree polynomial such that $v^\lambda \mid e$ and $\gcd(e/v^\lambda, v) = 1$. Let $k > \max\{1, \lambda\}$. Find $b_i, c_i \in C[x]$ to reduce the multiplicity k:

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Case 1. If $\lambda = 0$, then v is coprime with e and

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Fuchsian case (Chen, van Hoeij, Kauers, Koutschan 2017)

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Non-Fuchsian case (Chen, Du, Kauers, 2023)

D-finite functions: Hermite reduction at infinity

Example. Let $L = x^3 D^2 + (3x^2 + 2)D \in \mathbb{Q}(x)[D]$.

- After Hermite reduction at finite places, we get

$$f = \left(\frac{2x^2\omega_1 + 4x^2\omega_2}{3x^3}\right)' + \frac{(-4-3x^2)\omega_1 + (13-9x^2)\omega_2}{3x^2}$$

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- $\{\omega_1, \omega_2\} = \{1, x^3 D\}$ is also a local integral basis at infinity.
- Reducing the valuation at infinity (the degree in x) by Hermite reduction at infinity yields that

$$h = (-x\omega_1 - 3x\omega_2)' - \frac{4}{3x^2}\omega_1 - \frac{2}{3x^2}\omega_2$$

D-finite functions: symbolic integration

The remainder h of Hermite reduction at finite places is

$$f = g' + h = g' + \sum_{i=1}^{n} \frac{r_i}{d}\omega_i + \sum_{i=1}^{n} \frac{s_i}{e}\omega_i,$$

where $\deg(r_i) < \deg(d)$ and d is squarefree.

Remarks.

- An integral basis may not be a local integral basis at infinity. (Aldossari 2020: $\exists$ an integral basis that is normal at infinity.)

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- Creative telecoping: Let $f \in A = C(t,x)\langle D_t, D_x\rangle / I$. Find a nonzero $T \in C(t)\langle D_t\rangle$ and $g \in A$ such that $T(t, D_t)(f) = D_x(g)$.

D-finite functions: an example (continued)

Example. Let $L = x^3 D^2 + (3x^2 + 2)D \in \mathbb{Q}(x)[D]$.

- After Hermite reduction at finite places and at infinity, we get

$$f = \left(\left(\frac{2}{3x} - x \right) \omega_1 + \left(\frac{4}{3x} - 3x \right) \omega_2 \right)' - \frac{4}{3x^2}\omega_1 - \frac{2}{3x^2}\omega_2.$$

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$$U = \mathrm{span}_{\mathbb{Q}} \left\{ -\frac{2}{x^2} \omega_1 - \frac{1}{x^2} \omega_2,\ \frac{1}{x^3} \omega_2,\ -\frac{2}{x^3} \omega_2 \right\}.$$

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- So f is integrable in A:

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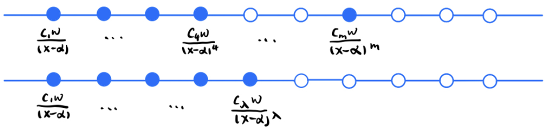
Comparison with other reduction algorithms

	Bostan et al.s Alg.	van der Hoeven's Alg.	Our Alg.
goal	reduce the pole orders at $(x-\alpha)$ and $\frac{1}{x}$ $f=\frac{aw}{(x-\alpha)^k}$	$\xrightarrow{-g'}$	$r=f-g'=\frac{cw}{(x-\alpha)^{k-1}}$
tools	Larange's identity + generalized H.R.	matrix & "good" bases + head/tail reductin	integral bases + H.R.
certificates	$g=L^*\left(\frac{b}{(x-\alpha)^{k-\tau_\alpha}}\right)$ $b\in\bar{C}(x)_\alpha$ → rational	$g=\frac{bw}{(x-\alpha)^{k-\tau_\alpha}}$ $b\in\bar{C}^n$ → D-finite	$g=\frac{(b_0+b_1(x-\alpha)+\cdots+b_{k-2}(x-\alpha)^{k-2})w}{(x-\alpha)^{k-1}}$ $b_i\in\bar{C}^n$ → D-finite
remainder of reductions	other Alg. Our Alg.		

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Main results.

- Hermite reduction for univariate D-finite functions
- new telescoping algorithm for bivariate D-finite functions

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- `https://github.com/LixinDu/HermiteReduction`

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Thank you!