Dynamics in Symbolic Integration and Summation

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Fundamental Theorem of Calculus

Newton–Leibniz Theorem. Let $f(x)$ be a continuous function on $[a,b]$ and let $F(x)$ be defined by

$$F(x) = \int_a^x f(t)dt \quad \text{for all } x \in [a,b].$$

Then $F(x)' = f(x)$ for all $x \in [a,b]$ and

$$\int_a^b f(x)\,dx = F(b) - F(a). \qquad \text{(Newton–Leibniz formula)}$$

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Definite Integration $\rightsquigarrow$ Indefinite Integration

$$\int_1^2 \log(x)\,dx = F(2) - F(1) = 2\log(2) - 1, \quad \text{where } F(x) = x\log(x) - x.$$

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Definite Integration $\rightsquigarrow$ Indefinite Integration

$$\int_0^{+\infty} \exp(-x^2)\,dx = ?$$

Differential Algebra

Differential Ring and Differential Field. Let R be an integral domain. An additive map $D : R \to R$ is called a derivation on R if

$$D(f \cdot g) = f \cdot D(g) + g \cdot D(f). \qquad \text{(Leibniz's rule)}$$

The pair (R, D) is called a differential ring. If R is a field, it is then called a differential field.

Example. Let $E := \mathbb{C}(x)(t_1, t_2, t_3, \ldots, t_n)$ with

$$t_1 = \sqrt{x^2+1}, \quad t_2 = \log(1+t_1^2), \quad t_3 = \exp\left(\frac{1+t_1}{t_1+t_2^2}\right), \ldots$$

Elementary Extensions

Differential Extension. (R^*, D^*) is called a differential extension of (R, D) if $R \subseteq R^*$ and $D^* |_R = D$.

Elementary Extension. Let (E, D) be a differential extension of (F, D). An element $t \in E$ is elementary over F if one of the following conditions holds:

- t is algebraic over F, i.e., $P(t) = 0$ for some $P \in F[z] \setminus \{0\}$;
- t is exponential over F, i.e., $D(t)/t = D(u)$ for some $u \in F$;
- t is logarithmic over F, i.e., $D(t) = D(u)/u$ for some $u \in F$.

Elementary Functions

Definition. An function f is elementary over $\mathbb{C}(x)$ if

$$f \in \mathbb{C}(x)(t_1, \ldots, t_n),$$

where t_i is elementary over $\mathbb{C}(x)(t_1, \ldots, t_{i-1})$ for all $i = 2, \ldots, n$.

Example.

$$f(x) = \frac{\pi}{\sqrt{\log\left(\exp\left(\sqrt{\frac{1}{3x^2+3x+1}}\right)^2 + x^2 + 1\right)}}$$

Then $f(x)$ is elementary since

$$f \in \mathbb{C}(x)(t_1, t_2, t_3, t_4),$$

where

$$t_1 = \sqrt{\frac{1}{3x^2+3x+1}}, \quad t_2 = \exp(t_1), \quad t_3 = \log(t_2^2 + x^2 + 1), \quad t_4 = \sqrt{t_3}.$$

Symbolic Integration

Let (F, D) and (E, D) be two differential fields such that $F \subseteq E$.

> Problem. Given $f \in F$, decide whether there exists $g \in E$ s.t. $f = D(g)$. If g exists, call f integrable in E.

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Elementary Integration Problem. Given an elementary function $f(x)$ over $\mathbb{C}(x)$, decide whether $\int f(x)dx$ is elementary or not.

Example. The following integrals are not elementary over $\mathbb{C}(x)$:

$$\int \exp(x^2)dx, \quad \int \frac{1}{\log(x)}dx, \quad \int \frac{\sin(x)}{x}dx, \quad \int \frac{dx}{\sqrt{x(x-1)(x-2)}}, \cdots$$

Symbolic Integration

Let (F, D) and (E, D) be two differential fields such that $F \subseteq E$.

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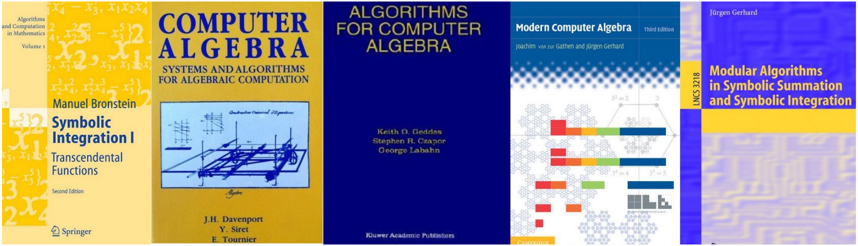
Selected books on Symbolic Integration:



Liouville's Theorem

Theorem (Liouville1835). Let $f(x)$ be elementary over $\mathbb{C}(x)$, i.e.,

$$f \in F = \mathbb{C}(x)(t_1, t_2, \ldots, t_n).$$

If $\int f(x)\,dx$ is elementary, then

$$\int f(x)\,dx = \underbrace{g_0}_{F\text{-part}} + \underbrace{\sum_{i=1}^{n} c_i \log(g_i)}_{\text{transcendental part}},$$

where $g_0, g_1, \ldots, g_n \in F$ and $c_1, \ldots, c_n \in \mathbb{C}$.

Remark. With the above theorem, Liouville proved that the integrals

$$\int \exp(x^2)\,dx, \quad \int \frac{1}{\log(x)}\,dx, \quad \int \frac{\sin(x)}{x}\,dx, \ldots$$

are not elementary.

Two classical theorems

Liouville-Hardy Theorem. Let $f \in \mathbb{C}(x)$. Then $f \cdot \log(x)$ is elementary integrable over $\mathbb{C}(x)$ if and only if

$$f = \frac{c}{x} + g' \quad \text{for some } c \in \mathbb{C} \text{ and } g \in \mathbb{C}(x).$$

Liouville's Theorem. Let $f, g \in \mathbb{C}(x)$. Then $f \cdot \exp(g)$ is elementary integrable over $\mathbb{C}(x)$ if and only if

$$f = h' + g'h \quad \text{for some } h \in \mathbb{C}(x).$$

Stability in dynamical systems

A (discrete) dynamical system is a pair (X,ϕ) with X being any set and $\phi: X \to X$ a self-map on X.

- Fixed points:

$$\mathrm{Fix}(\phi,X) = \{x \in X \mid \phi(x) = x\}.$$

- Periodic points:

$$\mathrm{Per}(\phi,X) = \{x \in X \mid \phi^n(x) = x \text{ for some } n \in \mathbb{N}\setminus\{0\}\}.$$

- Stable points:

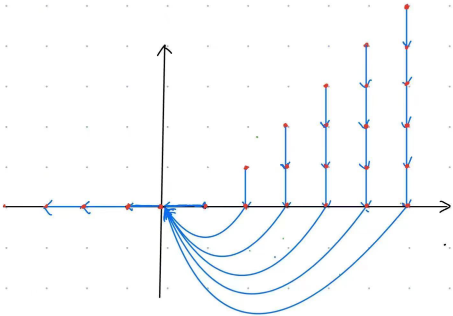
$$\mathrm{Stab}(\phi,X) = \{x \in X \mid \exists \{x_i\}_{i\geq 0} \text{ s.t. } x_0 = x \text{ and } \phi(x_{i+1}) = x_i \text{ for } i \in \mathbb{N}\}.$$

- Attractive points:

$$\mathrm{Attrac}(\phi,X) = \bigcap_{i\in\mathbb{N}} \phi^i(X).$$

$$\mathrm{Fix}(\phi,X) \subseteq \mathrm{Per}(\phi,X) \subseteq \mathrm{Stab}(\phi,X) \subseteq \mathrm{Attrac}(\phi,X).$$

Godelle's example



Example. Let $X = \{(i,j) \in \mathbb{Z}^2 \mid 0 \leq j \leq \max\{i-1,0\}\}$ and $\phi : X \to X$ be such that

$$\phi((i,j)) = (i,j-1) \text{ if } j > 0 \text{ and } \phi((i,0)) = (\min i-1,0,0).$$

Then $\mathrm{Stab}(\phi,X) = \emptyset$ and $\mathrm{Attrac}(\phi,X) = \{(i,0) \mid i \leq 0\}$.

Stability in differential fields

Idea. Viewing a differential field (K, D) as a dynamical system.

$$D(f+g) = D(f) + D(g) \quad \text{and} \quad D(fg) = gD(f) + fD(g).$$

Definition. $C_K := \{c \in K \mid D(c) = 0\}$ is called the constant subfield of (K, D).

Remark. K is a C_K-vector space and $D : K \to K$ is C_K-linear.

Proposition. Let (K, D) be a differential field of char. zero. Then

$$\mathrm{Stab}(D, K) = \mathrm{Attrac}(D, K).$$

Stability Problem. Given $f \in K$, decide whether f is stable or not, i.e., for all $i \in \mathbb{N}$, $f = D^i(g_i)$ for some $g_i \in K$.

Structure theorem

Lemma. Let (K, D) be a differential field with $D(x) = 1$. Then

$$f \in K \text{ is stable in } K$$

$$\Updownarrow$$

$$\text{for all } i \in \mathbb{N},\ x^i f = D(g_i) \text{ for some } g_i \in K$$

Theorem. Let (K, D) be a differential field with $D(x) = 1$. Then $\mathrm{Stab}(D, K)$ forms a differential $C_K[x]$-module.

Problem. Is $\mathrm{Stab}(D, K)$ always a free $C_K[x]$-module?

Example. $\exp(c \cdot x)$ is stable, so are

$$x^n \exp(c \cdot x), \quad x^n \sin(c \cdot x), \quad x^n \cos(c \cdot x), \quad \ldots$$

Integral used in the proof of the irrationality of π:

$$I_n(x) = \int_{-1}^{1} (1 - z^2)^n \cdot \cos(xz)\, dz \quad (n \in \mathbb{N})$$

Stable elementary functions

Theorem. Let $D = d/dx$ and $f, g \in \mathbb{C}(x)$ with $g \notin \mathbb{C}$. Then

(i) f is always stable in the field of elementary functions.

(ii) f is stable in $(\mathbb{C}(x), D)$ iff $f \in \mathbb{C}[x]$.

(iii) $f \cdot \log(x)$ is stable in $(\mathscr{E}_{\mathbb{C}(x)}, D)$ iff $f \in \mathbb{C}[x, x^{-1}]$.

(iv) $f \cdot \exp(g)$ is stable in $(\mathscr{E}_{\mathbb{C}(x)}, D)$ iff $f \in \mathbb{C}[x]$ and $g = ax + b$ with $a, b \in \mathbb{C}$ with $a \neq 0$.

Examples.

- Stable basic elementary functions: $f(x) \in \mathbb{C}(x)$, $\exp(ax+b)$,

  $$\log(x), \quad \sin(x), \quad \cos(x), \quad \arcsin(x) \quad \arccos(x), \quad \arctan(x), \ldots$$

- Non-stable basic elementary functions:

  $$\tan(x), \cot(x), \sec(x), \csc(x), \ldots$$

D-finite power series

Definition. A series $f \in \mathbb{C}[[x]]$ is D-finite over $\mathbb{C}(x)$ if it satisfies

$$a_r(x) \cdot D_x^r(f) + \cdots + a_1 \cdot D_x(f) + a_0 \cdot f = 0,$$

where $a_i \in \mathbb{C}[x]$ and $a_r \neq 0$. Equivalently,

$$\dim_{\mathbb{C}(x)} \left(\text{span}_{\mathbb{C}(x)}\{D_x^i(f) \mid i \in \mathbb{N}\}\right) < +\infty$$

- R. P. Stanley. Differentiably Finite Power Series. *European Journal of Combinatorics*, 1: 175–188, 1980.
- L. Lipshitz. D-Finite Power Series. *Journal of Algebra*, 122: 353–373, 1989.
- M. Kauers. D-Finite Functions. *Springer*, 2023, 602 pages.

Exact integration

Definition. Let $f \in \mathbb{C}[[x]]$ be D-finite with

$$p_d \cdot D_x^d(f) + p_{d-1} \cdot D_x^{d-1}(f) + \cdots + p_0 \cdot f = 0.$$

If d is minimal, then call d the order of f, denoted by $\operatorname{ord}(f)$.

Remark. In general, the formal integral int(f):= $\int f(x)dx$ has the minimal annihilator of order $\operatorname{ord}(f) + 1$.

Exact Integration. In 1997, Abramov and van Hoeij gave an algorithm to decide whether $\int f(x)dx$ has an annihilator of the same order as that of f.

Stable D-finite power series

Let $f(x) \in \mathbb{C}[[x]]$ be a D-finite power series.

Definition. $f(x)$ is stable if $\exists \{g_i\}_{i\in\mathbb{N}} \in \mathbb{C}[[x]]$ s.t. $g_0 = f$ and

$$g_i = D_x(g_{i+1}) \text{ and } \mathrm{ord}(g_i) = \mathrm{ord}(f) \quad \text{for all } i \in \mathbb{N}.$$

$f(x)$ is eventually stable if $\exists\, m \in \mathbb{N}$ s.t. $\mathrm{int}^m(f)$ is stable.

Theorem. Any D-finite power series is eventually stable.

Example. The Airy function $\mathrm{Ai}(x)$ satisfies

$$y''(x) = xy(x).$$

By Abramov-van Hoeij's algorithm, we have $\mathrm{Ai}(x)$ is not stable, but is eventually stable with $\mathrm{ord}(\mathrm{int}^m(\mathrm{Ai}(x))) = 3$ for all $m \geq 2$.

Stability index

Definition. For any $P \in \mathbb{C}[x]\langle D_x \rangle$, there exist a nonzero polynomial $\xi_P(z) \in \mathbb{C}[z]$ and an integer σ_P such that for any $s \in \mathbb{Z}$,

$$P(x^s) = \xi_P(s) x^{s+\sigma_P} (1 + c_1 x^{-1} + c_2 x^{-2} + \cdots)$$

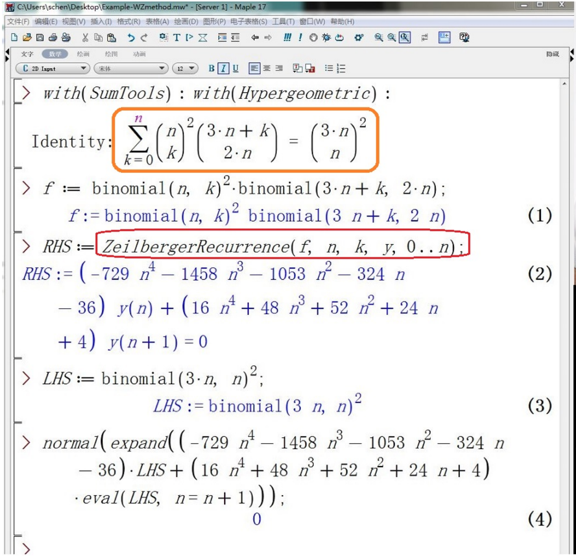
where $c_i \in \mathbb{C}$. The polynomial $\xi_P(z)$ is called the indicial polynomial of P at ∞.

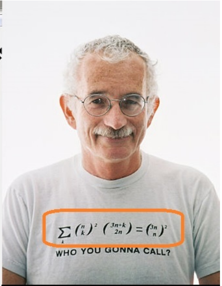
Theorem. Let $f \in \mathbb{C}[[x]]$ be D-finite with minimal annihilator $L \in \mathbb{C}(x)\langle D_x \rangle$. Let $p \in \mathbb{C}[x]$ be the polynomial of minimal degree such that $pL \in \mathbb{C}[x]\langle D_x \rangle$ and M be the maximal nonnegative integer root of $\xi_L(z)$. Let

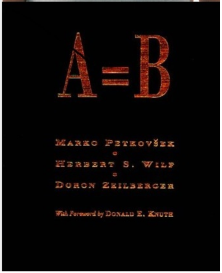
$$\Omega(L) := \max\{0, M + \sigma_L + \deg(p) + 1\}.$$

Then $Int^{\Omega(L)}(f))$ is stable.

Symbolic summation on T-shirt







Telescoping

Problem. For a sequence $f(k)$ in some class $\mathfrak{S}(k)$, decide whether there exists $g(k) \in \mathfrak{S}(k)$ s.t.

$$f(k) = g(k+1) - g(k) = \Delta_k(g)$$

$$\Downarrow$$

$$\sum_{k=a}^{b} f(k) = g(b+1) - g(a)$$

Examples.

- Rational sums

$$\sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum_{k=1}^{n} \Delta_k\left(-\frac{1}{k}\right) = 1 - \frac{1}{n+1}$$

- Hypergeometric sums

$$\sum_{k=0}^{n} \frac{\binom{2k}{k}^2}{(k+1)4^{2k}} = \sum_{k=0}^{n} \Delta_k\left(\frac{4k\binom{2k}{k}^2}{4^{2k}}\right) = \frac{4(n+1)\binom{2n+2}{n+1}^2}{4^{2n+2}}$$

Gosper's algorithm

A sequence $H(k)$ is hypergeometric if

$$\frac{H(k+1)}{H(k)} \in \mathbb{C}(k).$$

In 1978, Gosper solved the telescoping problem for hypergeometric terms.

Proc. Natl. Acad. Sci. USA
Vol. 75, No. 1, pp. 40–42, January 1978
Mathematics

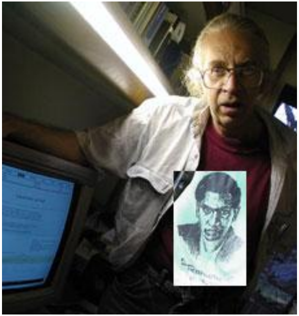
Decision procedure for indefinite hypergeometric summation

(algorithm/binomial coefficient identities/closed form/symbolic computation/linear recurrences)

R. WILLIAM GOSPER, JR.

Input: A hypergeometric $H(k)$
Output: A hypergeometric $G(k)$ if

$$H = \Delta_k(G)$$

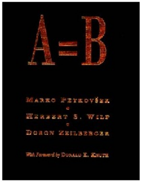


Bill Gosper

Stable hypergeometric sequences

An identity from the book **A=B**:

$$\sum_{n_s=0}^{n} \sum_{n_{s-1}=0}^{n_s} \cdots \sum_{n_1=0}^{n_2} \frac{\binom{2n_1}{n_1}}{4^{n_1}} = \frac{(2n+2s-1)!!}{(2n-1)!!(2s-1)!!} \frac{\binom{2n}{n}}{4^n} = \frac{\binom{2n+2s}{2s}}{\binom{n+s}{s}} \frac{\binom{2n}{n}}{4^n}$$



Problem. Classifying iteratively summable (stable) hypergeometric sequences.

Classification Theorem. A hypergeometric $H(k)$ is stable iff $H(k)$ is

- Exp-polynomial: $p(k) \cdot \alpha^k$ with $p \in \mathbb{C}[k], \alpha \in \mathbb{C} \setminus \{0\}$ or
- Gamma-polynomial: $p(k) \cdot \frac{\Gamma(k+\alpha)}{\Gamma(k+\beta)}$ with $p \in \mathbb{C}[k], \alpha, \beta \in \mathbb{C}$ and $\alpha - \beta \notin \mathbb{Z}$.

P-recursive sequences

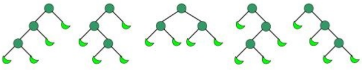
Definition. A sequence $s : \mathbb{N} \to K$ is P-recursive over K if it satisfies

$$p_d \cdot s(n+d) + p_{d-1} \cdot s(n+d-1) + \cdots + p_0 \cdot s(n) = 0,$$

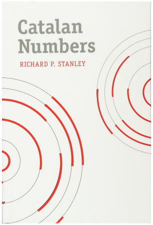
where $p_i \in K[n]$ and $p_d \cdot p_0 \neq 0$.

Example. The Catalan numbers $C(n) = \frac{1}{n+1}\binom{2n}{n}$ satisfy the relation

$$(n+2)C(n+1) - (4n+2)C(n) = 0, \quad \text{with } C(0) = 1.$$



C(3) = 5



Stability in difference fields

Idea. Viewing a difference field (K, Δ) as a dynamical system.

$$\Delta(f+g) = \Delta(f) + \Delta(g) \quad \text{and} \quad \Delta(fg) = \sigma(f)\Delta(g) + g\Delta(f).$$

Remark. Let $C_K := \{c \in K \mid \Delta(c) = 0\}$. Then K is a C_K-vector space and $\Delta : K \to K$ is C_K-linear.

Proposition. Let (K, Δ) be a difference field of char. zero. Then

$$\text{Stab}(\Delta, K) = \text{Attrac}(\Delta, K).$$

Stability Problem. Given $f \in K$, decide whether f is stable or not, i.e., for all $i \in \mathbb{N}$, $f = \Delta^i(g_i)$ for some $g_i \in K$.

Exact Summation

Definition. Let $a(n)$ be a P-recursive sequence

$$p_d \cdot a(n+d) + p_{d-1} \cdot a(n+d-1) + \cdots + p_0 \cdot a(n) = 0.$$

If d is minimal, then call d the order of $a(n)$, denoted by $\operatorname{ord}(a(n))$.

Remark. In general, the indefinite sum

$$s(n) = a(1) + a(2) + \cdots + a(n),$$

satisfies a linear recurrence of order $\operatorname{ord}(a) + 1$.

Exact Summation. In 1997, Abramov and van Hoeij gave an algorithm to decide whether $\operatorname{ord}(s(n)) = \operatorname{ord}(a(n))$.

Stable P-recursive sequences

Let $a(n)$ be a P-recursive sequence.

Definition. $a(n)$ is stable if $\exists\ \{g_i\}_{i\in\mathbb{N}} \in S/I$ s.t. $g_0 = a(n)$ and

$$g_i = \Delta(g_{i+1}) \text{ and } \operatorname{ord}(g_i) = \operatorname{ord}(a(n)) \quad \text{for all } i \in \mathbb{N}.$$

$a(n)$ is eventually stable if $\exists\ m \in \mathbb{N}$ s.t. $\sum^m (a(n))$ is stable.

Theorem. Any P-recursive sequence is eventually stable.

Example. Let $a(n) = 1/n$ and $H(n) = \sum_{i=1}^{n-1} a(i)$ with $\Delta(H) = a$. We have

$$(n+1)a(n+1) - na(n) = 0.$$

$$(n+1)H(n+2) - (2n+1)H(n+1) + nH(n) = 0.$$

By Abramov-van Hoeij's algorithm, we have $a(n)$ is not stable, but is eventually stable at order 2.

Open problems

Problem. Characterizing stable algebraic functions in $(\overline{\mathbb{C}(x)}, d/dx)$.

Problem. Characterizing stable elementary functions over $\mathbb{C}(x)$.

Conjecture. Let $f(x)$ be an elementary function over $\mathbb{C}(x)$. Then

$$\{i \in \mathbb{N} \mid x^i \cdot f(x) \text{ is elementary integrable over } \mathbb{C}(x)\}$$

is a union of finitely many arithmetic progressions.

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Thank You!