An Invitation to Symbolic Dynamics on Groups

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DART XI

Tuesday 6 June 2023

AMENABLE GROUPS

G group is AMENABLE if, equivalently

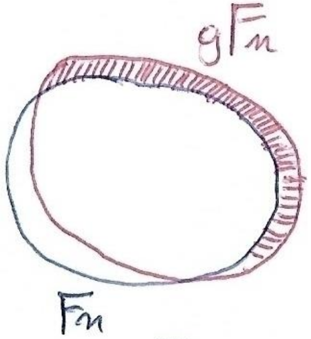
(vN) $\exists \ \mu : \{0,1\}^G \longrightarrow [0,1]$

- $\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B)$
- $\mu(G) = 1$ $\quad \forall A, B \subseteq G$
- $\mu(gA) = \mu(A)$ $\quad \forall g \in G$

e.g. FINITE GROUPS

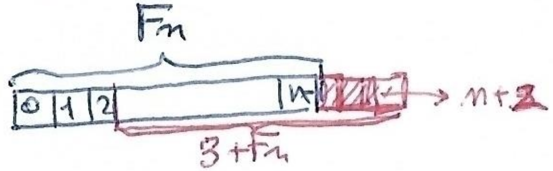
(Følner) $\exists \ (F_n)_{n \in \mathbb{N}}$, $F_n \subset G$ finite

s.t. $$\lim_{n \to \infty} \frac{|gF_n \setminus F_n|}{|F_n|} = 0 \quad \forall g \in G$$



e.g. $\mathbb{Z}^d$, $d \geq 1$. $(F_n = \{0, 1, \dots, n-1\})$

$|(g + F_n) \setminus F_n| = |g|$



(Kesten–Day) for finitely generated groups: $\|M\| = 1$

M: Markov operator on $\ell^2(G)$

(Tarski) $\nexists$ PARADOXICAL DECOMPOSITION:

$$G = A_1 \sqcup A_2 \sqcup \cdots \sqcup A_m \sqcup B_1 \sqcup B_2 \sqcup \cdots \sqcup B_n$$

$$(*) \quad = g_1 A_1 \sqcup g_2 A_2 \sqcup \cdots \sqcup g_m A_m$$

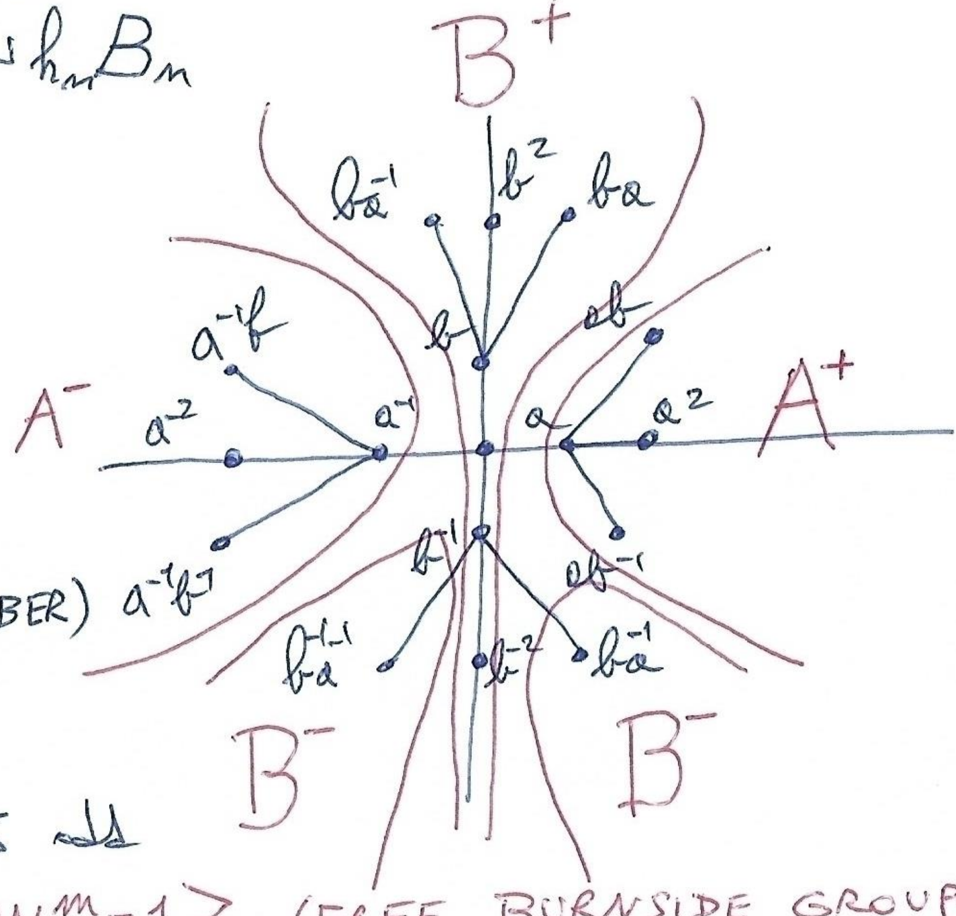
$$= h_1 B_1 \sqcup h_2 B_2 \sqcup \cdots \sqcup h_n B_n$$

FREE GROUP ON 2 GENERATORS (a, b)

$$F_2 = A^+ \sqcup A^- \sqcup B^+ \sqcup B^-$$

$$= A^+ \sqcup a A^-$$

$$= B^+ \sqcup b B^-$$



$$T(G) = \begin{cases} \min(m+n) : \exists \text{ PD as in } (*) \\ \infty \quad \nexists \text{ PD} \end{cases}$$

(TARSKI NUMBER)

- $T(G) = 4 \iff G \geq F_2$
- $6 \leq T(B(2,m)) \leq 14$, $m \geq 665$ odd

$B(n,m) = \langle x_1, x_2, \ldots, x_n : w^m = 1 \rangle$ (FREE BURNSIDE GROUPS)

CLOSURE PROPERIES

- subgroups
- quotients
- extensions $1 \to N \to G \to H \to 1$
- direct limits

EXAMPLES

- Abelian, nilpotent, solvable
- subexponential growth

NONEXAMPLES

- $G \geqslant F_2$
- TARSKI MONSTERS (Olshansky)
  FREE BURNSIDE GROUPS (Adyan)

CELLULAR AUTOMATA

- A set, G group $\rightsquigarrow A^G = \prod_{g\in G} A = \{x : G \to A\}$
  alphabet, universe, configurations

- $G \curvearrowright A^G$: $(gx)(h) = x(g^{-1}h)$ G-shift

- $\tau : A^G \longrightarrow B^G$ is a CELLULAR AUTOMATON
  if $\exists\, M \subseteq G$ finite memory, $\mu : A^M \longrightarrow B$ local defining map s.t.

$$\tau(x)(g) = \mu\left((g^{-1}x)\big|_M\right) \quad \forall g\in G,\ \forall x\in A^G$$

EX 1 $G=\mathbb{Z}$, $M=\{0,1,-1\}$
$A=\{0,1\}$; μ = MAJORITY

$x = \cdots |1|1|\boxed{0|1|0}|0|1| \cdots$

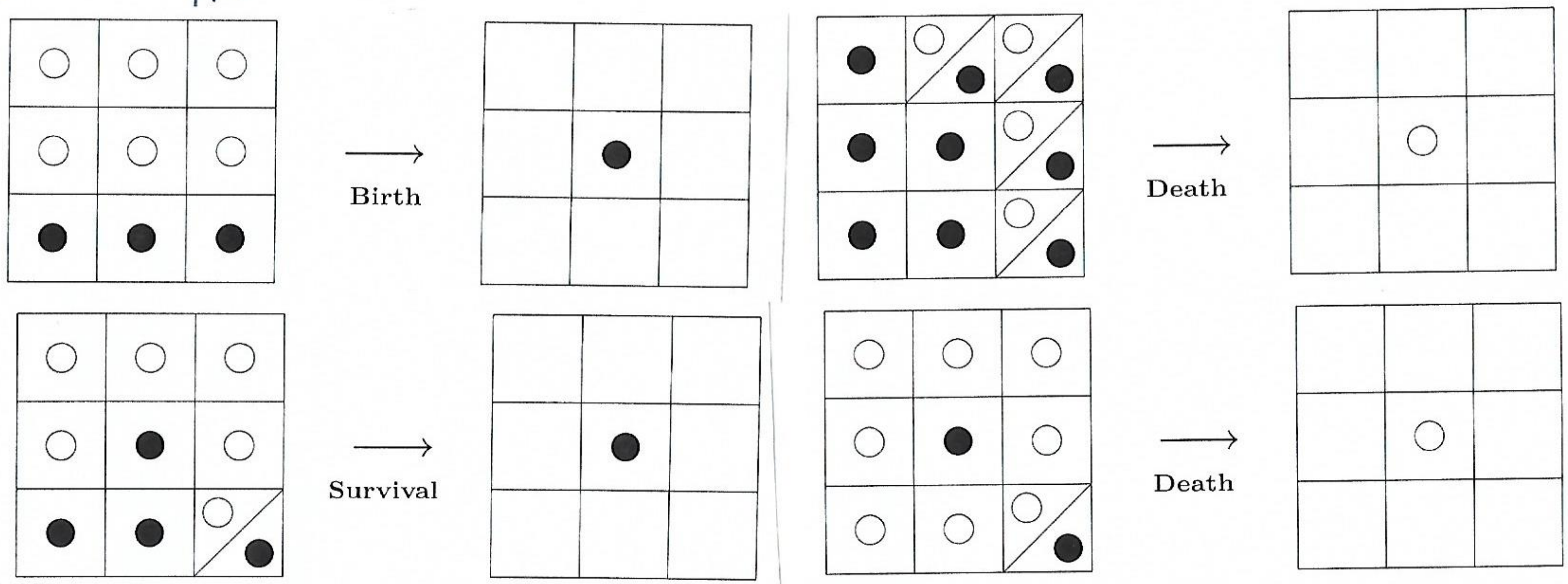
MAJORITY

$\tau(x) = \cdots |1|1|1|\boxed{0}|0|0|?|$

EX 2 (Conway's GAME OF LIFE), $G=\mathbb{Z}^2$, $M=\{-1,0,1\}^2$

μ is described $\rightsquigarrow$

$A = \{\circ, \bullet\}$
dead cell, live cell



Equip A^G with the <u>PRODISCRETE UNIFORM STRUCTURE</u>

(a BASE of ENTOURAGES is constituted by $W_F = \{(x,y) \in A^G \times A^G : x|_F = y|_F\}$, $F \underset{\text{finite}}{\subseteq} G$)

<u>THEOREM</u> (CS-Coornaert)

$\tau : A^G \to B^G$ is a CELLULAR AUTOMATON $\iff$

- τ is <u>UNIFORMLY CONTINUOUS</u> w.r. to the prodiscrete unif. structures on A^G & B^G
- τ is G-equivariant $\left(g\tau(x) = \tau(gx) \quad \begin{matrix} \forall g \in G \\ \forall x \in A^G \end{matrix} \right)$

THE GARDEN OF EDEN THEOREM

$x, y \in A^G$, $x \sim y$ (homoclinic relation), if $\{g \in G : x(g) \neq y(g)\}$ is FINITE

τ is PREINJECTIVE if $\begin{cases} \tau(x) = \tau(y) \\ x \sim y \end{cases} \Longrightarrow x = y.$

THEOREM (CS-Machì-Scarabotti, 1999)

GARDEN OF EDEN

1) $|A| < \infty$, G amenable, $\tau : A^G \to A^G$ cellular automaton: $\boxed{\text{preinjective} \Longleftrightarrow \text{surjective.}}$

2) If $\boxed{G \geqslant F_2}$ then both implications fail in general.

THEOREM (Bartholdi, 2010 & 2019)

If G NONAMENABLE then both implications fail.

$X \subseteq A^G$ (G amenable)

$$\mathrm{ent}_{\mathcal{F}}(X) = \limsup_{n \to \infty} \frac{\log |X_{F_n}|}{|F_n|}$$

where $X_{F_n} = \{x|_{F_n} : x \in X\} \subseteq A^{F_n}$.

$\Longleftrightarrow$ $\boxed{\mathrm{ent}_{\mathcal{F}}(\tau(A^G)) = \mathrm{ent}_{\mathcal{F}}(A^G)}$ $(= \log |A|)$

Ornstein-Weiss. $X \subseteq A^G$, G-inv

- $\mathrm{ent}_{\mathcal{F}}$ true limit
- indep of $\mathcal{F} = (F_n)_{n \in \mathbb{N}}$.

GoE etymology:

$$A^G \xrightarrow{\tau} A^G \xrightarrow{\tau} A^G \xrightarrow{\tau} \cdots$$

$$x \longmapsto \tau(x) \longmapsto \tau^2(x) \longmapsto \cdot$$

GOTTSCHALK CONJECTURE

$|A|<\infty$, $\tau : A^G \to A^G$ $\boxed{\text{injective}}$ cellular automaton $\Rightarrow$ $\boxed{\text{surjective}}$

(SURJUNCTIVE group)

- RESIDUALLY FINITE groups (Lawton)

$\boxed{* \ \forall g \in G \setminus \{1_G\} \ \exists F \text{ finite group } \& \ \varphi : G \xrightarrow{\text{homo}} F \text{ s.t. } \varphi(g) \neq 1_F,}$

$** \ \bigcap_{\substack{H \leq G \\ [G:H]<\infty}} H = \{1_G\}$, $***$ $\forall K \underset{\text{finite}}{\subset} G \ \exists F$ finite group $\&$ $\varphi : G \xrightarrow{\text{homo}} F$ s.t. $\varphi|_K$ is INJECTIVE

$**** \ \mathcal{P}er(A^G) = \{x \in A^G : Gx \text{ is finite}\} \left(= \{x \in A^G : [G : Stab_G(x)] < \infty\} \right)$ is DENSE in A^G.

$g \in G$ s.t. $gx = x$

- AMENABLE groups (GOE theorem)
- SOFIC groups (Gromov-Weiss)

$* \ \exists (\varphi_n, k_n)_{n\in\mathbb{N}}$, $\varphi_n : G \to Sym(k_n)$

- $d_n^H(\varphi_n(g_1 g_2), \varphi_n(g_1)\varphi_n(g_2)) \xrightarrow[n\to\infty]{} 0$ $\forall g_1, g_2 \in G$
- $d_n^H(\varphi_n(g_1), \varphi_n(g_2)) \xrightarrow[n\to\infty]{} 1$ $g_1 \neq g_2$

$d_n^H : Sym(k_n) \times Sym(k_n) \to [0,1]$

NORMALIZED Hamming distance

LINEAR CELLULAR AUTOMATA

K field, A, B vector space$/K$, G group $\rightsquigarrow$ A^G vector space$/K$

$\tau : A^G \to B^G$ is called LINEAR cellular automaton if it is K-linear ($\iff \mu : A^M \to B$ is K-linear).

($x \sim y$ if $x - y \in A[G] = \{x \in A^G : x(g) = 0_A \ \forall g \in G$ except finitely many$\}$)

EX 3 (DISCRETE LAPLACIAN) $K = \mathbb{C}$, $S \subseteq G$ finite subset $\neq \emptyset$

$$\boxed{\Delta(x)(g) = x(g) - \frac{1}{|S|}\sum_{s \in S} x(gs)}$$

$\forall g \in G$
$\forall x \in A^G$

1 THM (CS-Coornaert) $\tau : A^G \to A^G$ linear-cellular automaton

- $\dim_K A < \infty$, $\boxed{G \text{ amenable}}$,
- τ preinjective $\iff$ τ surjective

$\boxed{\tau|_{A[G]} \text{ injective}}$ $\Leftarrow$ $\mathrm{mdim}_{\mathcal{F}}(\tau(A^G)) = \dim_K A$

$$\boxed{\mathrm{mdim}_{\mathcal{F}}(X) = \limsup_{n\to\infty} \frac{\dim_K(X_{F_n})}{|F_n|}}$$

$X \subseteq A^G$ vector subspace

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2 <u>THM</u> (CS-Coornaert)

dim$_K A < \infty$, $\boxed{G \text{ sofic}}$, $\tau : A^G \to A^G$ $\boxed{\text{INJECTIVE}}$ linear cellular automaton $\Longrightarrow$ τ is $\boxed{\text{SURJECTIVE}}$.

<u>THM</u> (CS-Coornaert)

$d = \dim_K A < \infty$

K-algebra $LCA(G;A) \cong Mat_d(K[G])$

<u>COROLLARY</u> (ELEK-SZABÓ)

$\dim_K A < \infty$, G sofic $\Rightarrow K[G]$ is STABLY FINITE. (Kaplansky conjecture)

$$\boxed{\begin{array}{l} R \text{ is SF if } \forall d \geq 0, \ \forall A, B \in Mat_d(R) \\ \qquad AB = I_d \Rightarrow BA = I_d. \end{array}}$$

SUBSHIFTS

$X \subseteq A^G$ closed & G-invariant is called a SUBSHIFT.

- $\Omega \underset{\text{finite}}{\subseteq} G$, $p \in A^\Omega$ is a PATTERN
- $X \subseteq A^G$ subshift $\iff \exists\ \mathcal{F} \subseteq \{A^\Omega : \Omega \underset{\text{finite}}{\subseteq} G\}$ s.t.

DEFINING SET of FORBIDDEN PATTERNS

$$X = \{x \in A^G : (gx)|_\Omega \notin \mathcal{F}, \forall g \in G, \forall \Omega \underset{\text{finite}}{\subseteq} G\}$$

$\boxed{|A| < \infty}$

- if $\exists\ \mathcal{F}$ finite one says that X is of FINITE TYPE

EX 1 $X \subseteq \{0,1\}^{\mathbb{Z}}$, $x \in X$ s.t. $x(n)x(n+1) \neq 11$ $(\mathcal{F} = \{11\} \subseteq \{0,1\}^*)$

GOLDEN MEAN SUBSHIFT

EX 2 $X \subseteq \{0,1\}^{\mathbb{Z}}$, $x \in X$ s.t. $x(n)x(n+1)\cdots x(n+m+1) = 10^m1 \Rightarrow$ me

$(\mathcal{F} = \{10^{2k+1}1 : k \in \mathbb{N}\})$

EVEN SHIFT

$\boxed{|A|<\infty}$

- $X \subseteq A^G$ is called $\boxed{\text{SOFIC}}$ if $\exists B$ finite, $\tau : B^G \to A^G$ cellular automaton, $Y \subseteq B^G$ subshift of finite type s.t.

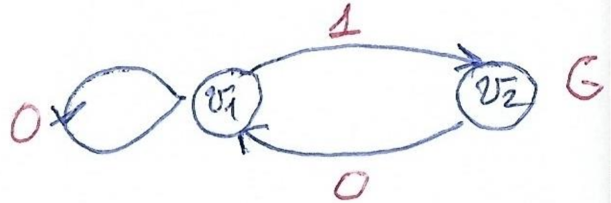
$$X = \tau(Y).$$

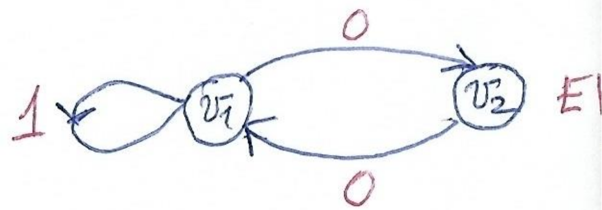
<u>EXAMPLE</u> $A = B = \{0,1\}$: $\tau : B^G \to A^G$ $M = \{0,1\}$ $\mu : \begin{cases} 00 \mapsto 1 \\ 01 \mapsto 0 \\ 10 \mapsto 0 \\ 11 \mapsto 0 \end{cases} \Longrightarrow X_{\text{EVEN}} = \tau(X_{\text{GM}}$

- $\mathcal{G} = (V, E, A)$ <u>finite</u> A-labeled graph

V = vertices, $E \subseteq V \times A \times V$ edges, A alphabet

$e = (v^-, a, v^+) \in E$ $\quad \alpha(e) = v^-$, $\omega(e) = v^+$, $\lambda(e) = a$





$\pi = (e_n)_{n \in \mathbb{Z}}$ <u>path</u> if $\omega(e_n) = \alpha(e_{n+1})$ $\forall n$

$[\lambda(\pi)](n) = \lambda(e_n)$.

<u>THM</u> $X \subseteq A^{\mathbb{Z}}$ is SOFIC $\iff \exists \mathcal{G}$ s.t. $X = \{\lambda(\pi) : \pi \text{ path in } \mathcal{G}\}$

$= X(\mathcal{G})$

GOE Thms & SURJUNCTIVITY for SUBSHIFTS

- FIORENZI: $|A| < \infty$, G amenable, $X \subseteq A^G$ of FINITE TYPE + STRONGLY IRREDUCIBLE

  $\tau : A^G \to A^G$ cellular automaton s.t. $\tau(X) \subseteq X$, then

  $\tau|_X$ is pre-inj $\Longleftrightarrow \tau(X) = X$ (GOE)

- FIORENZI: $|A| < \infty$, $X \subseteq A^{\mathbb{Z}}$ of FINITE TYPE + $\boxed{\text{IRREDUCIBLE}}$, $\tau(X) \subseteq X$

  $\tau|_X$ preinjective $\Longleftrightarrow \tau(X) = X$ (GOE)

  COUNTEREXAMPE FOR SOFIC SUBSHIFS

$L(X) = \{x(n)x(n+1)\cdots x(n+\ldots) : n \in \mathbb{Z}, m \in \mathbb{N}\ldots$

Language of X

X irred if $\forall u, v \in L(X)$ $\exists w \in A^*$ s.t. $uwv \in \ldots$

$\Updownarrow$ if X SOFIC

$\exists \mathcal{G}$ CONNECTED s.t. $X = X(\mathcal{G})$

- CS-Coornaert: $|A| < \infty$, G amenable, $X \subseteq A^G$ STRONGLY IRREDUCIBLE

  $\tau : A^G \to A^G$ cellular automata s.t. $\tau(X) \subseteq X$.

  $\tau|_X$ injective $\Longrightarrow \tau(X) = X$ (Surj)

- CS-Coornaert: $|A| < \infty$, $X \subseteq A^{\mathbb{Z}}$ SOFIC + IRREDUCIBLE

  $\tau|_X$ injective $\Longrightarrow \tau(X) = X$ (Surj)

+ LINEAR VERSIONS ...