Algebraic relations between solutions of differential equations.

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An example : the first Painlevé equation

$$P_1 : \frac{d^2y}{dx^2} = 6y^2 + x$$

Theorem (K. Nishioka - 2004)

Let $y_1, \ldots, y_n$ be solutions of P_1.

If $\mathrm{tr.deg.}_{\mathbb{C}(x)}\mathbb{C}(x)(y_1, y_1', \ldots y_n, y_n') < 2n$ then $\exists\ i < j$ such that $y_i = y_j$

There is no subvariety of $\mathbb{C}^{1+2n}$ invariant by

$$\frac{\partial}{\partial x} + \sum_{i=1}^{n} z_i \frac{\partial}{\partial y_i} + (6y_i^2 + x)\frac{\partial}{\partial z_i}$$

but the diagonals.

An example : the first Painlevé equation

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Theorem (K. Nishioka - 2004)

Let $y_1, \ldots, y_n$ be solutions of P_1 and $\mathbb{C}(x) \subset K$ be a differential extension

If $\mathrm{tr.deg.}_K K(y_1, y_1', \ldots y_n, y_n') < 2n$ then

1. *$\exists\ i$ such that $y_i \in K^{\mathrm{alg}}$ or*
2. *$\exists\ i < j$ such that $y_i = y_j$*

The problem for other Painlevé equations, $P_N(\alpha)$

- $P_2(\alpha) : \dfrac{d^2y}{dx^2} = 2y^3 + xy + \alpha$
- $P_3(\alpha) : \dfrac{d^2y}{dx^2} = \dfrac{1}{y}\left(\dfrac{dy}{dx}\right)^2 - \dfrac{1}{x}\dfrac{dy}{dx} + \dfrac{\alpha_1 y^2 + \alpha_2}{x} + \alpha_3 y^3 + \dfrac{\alpha_4}{y}$
- $P_4(\alpha), \quad P_5(\alpha), \quad P_6(\alpha)$

To give, for $N \in \{2, \ldots, 6\}$, the parameters α such that n distinct solutions of $P_N(\alpha)$ are mutually independent:

$$\mathrm{tr.deg.}_{\mathbb{C}(x)}\mathbb{C}(x)(y_1, y_1', \ldots y_n, y_n') = \sum \mathrm{tr.deg.}_{\mathbb{C}(x)}\mathbb{C}(x)(y_i, y_i')$$

Theorem (J. Nagloo & A. Pillay - 2017)

Let $N \in \{2, \dots 6\}$ *and* α *algebraically independent over* $\mathbb{Q}$*.*
If $y_1, \dots y_n$ *are solutions of* $P_N(\alpha)$ *such that*

$$\mathrm{tr.deg.}_{\mathbb{C}(x)}\mathbb{C}(x)(y_1, y_1', \dots y_n, y_n') < 2n$$

then

1. $\exists$ i *such that* $\mathrm{tr.deg.}_{\mathbb{C}(x)}\mathbb{C}(x)(y_i, y_i') < 2$ *or*
2. $\exists$ $i < j$ *such that* $\mathbb{C}(x, y_i, y_i')^{\mathrm{alg}} = \mathbb{C}(x, y_j, y_j')^{\mathrm{alg}}$

An analogy with groups

Corollary (of Goursat's Lemma)

Let G_i, $i = 1 \ldots n$, be simple groups and $H \subsetneq G_1 \times \ldots \times G_n$ a subgroup.

Then

- *$\exists\ i$ such that $pr_i : H \to G_i$ is not onto, or*
- *$\exists\ i < j$ and an isomorphism $\phi : G_i \to G_j$ such that the image of $pr_{i,j} : H \to G_i \times G_j$ is the graph of ϕ.*

We want to generalize Nagloo-Pillay Theorem replacing in the proof Model Theory by Group Theory.

We need to use differential Galois theory.

The general framework

The differential equations

$$\left.\begin{array}{ll} X_1: & y_1^{(m_1)} = E_1\left(x, y_1, \ldots, y_1^{(m_1-1)}\right), \\ & \vdots \\ X_n: & y_n^{(m_n)} = E_n\left(x, y_n, \ldots, y_n^{(m_n-1)}\right). \end{array}\right\} X^{[n]}$$

are seen as rational vector fields on the phase spaces

- $M_1 = \mathbb{C}^{1+m_1}$ with coordinates $x, y_1, \ldots, y_1^{(m_1-1)}$

$\vdots$

- $M^{[n]} = \mathbb{C}^{1+\sum m_i}$ with coord's $x, y_1, \ldots, y_1^{(m_1-1)}, y_2, \ldots, \ldots y_n^{(m_n-1)}$

We want to understand $X^{[n]}$-invariant algebraic subvarieties of $M^{[n]}$ under some hypothesis on each X_i.

The Theorem

$$\left.\begin{array}{ll} X_1: & y_1^{(m_1)} = E_1\left(x, y_1, \ldots, y_1^{(m_1-1)}\right), \\ & \vdots \\ X_n: & y_n^{(m_n)} = E_n\left(x, y_n, \ldots, y_n^{(m_n-1)}\right). \end{array}\right\} X^{[n]}$$

Assume the Galois groupoids of X_i are big enough.
If a solution $y(x)$ of $X^{[n]}$ satisfies

$$\mathrm{tr.deg.}_{\mathbb{C}(x)}\mathbb{C}(x)(y_1, \ldots, y_1^{(m_1-1)}, \ldots, y_n, \ldots, y_n^{(m_n-1)}) < \sum m_i$$

then

- $\exists\ i$ s.t. $\mathrm{tr.deg.}_{\mathbb{C}(x)}\mathbb{C}(x)(y_i, \ldots, y_i^{(m_i-1)}) < m_i$, or
- $\exists\ i < j$ s.t. $\mathbb{C}(x, y_i, \ldots, y_i^{(m_i-1)})^{\mathrm{alg}} = \mathbb{C}(x, y_j, \ldots, y_j^{(m_j-1)})^{\mathrm{alg}}$

Malgrange pseudogroup

- M an alg. variety over $\mathbb{C}$, $\dim M = m$, $\mathbb{C}(M)$ the field of rational functions.
- X a rational vector field on M.

Differential invariants of X

- $\partial_1, \ldots, \partial_m$ some symbols and $\mathbb{C}(M)_\infty$ the ∂-differential field generated by $\mathbb{C}(M)$.
- X_∞ the extension of X commuting with ∂s
- $\mathrm{Inv}(X) = \{H \in \mathbb{C}(M)_\infty : X_\infty \cdot H = 0\}$

Example $\quad M = \mathbb{C}^m, \quad \mathbb{C}(M) = \mathbb{C}(x_1, \ldots, x_m), \quad X = \sum a_i(x)\frac{\partial}{\partial x_i}$

- $\mathbb{C}(M)_\infty = \mathbb{C}(x_{i,\alpha} | i = 1, \ldots m; \alpha \in \mathbb{N}^m)$; $\partial_j(x_{i,\alpha}) = x_{i,\alpha+1_j}$
- $X_\infty = \sum_{i,\alpha} \partial^\alpha(a_i)\frac{\partial}{\partial x_{i,\alpha}}$
- $\mathcal{L}_X(\sum w_i(x)dx_i) = 0$ if and only if $\forall j,\ X_\infty \cdot \left(\sum w_i(x)x_{i,1_j}\right) = 0$.

For $\varphi : \mathcal{U} \to \mathcal{V}$ a biholomorphism between open subsets of M, $\varphi^*_\infty : Mer(\mathcal{V})_\infty \to Mer(\mathcal{U})_\infty$ is the morphism extending φ^* and commuting with ∂s.

Definition

$$\mathrm{Mal}(X) = \{\varphi \mid \forall H \in \mathrm{Inv}(X)\ \varphi^*_\infty(H) = H\}$$

Example If $\mathcal{L}_X\omega = 0$ then $\forall \varphi \in \mathrm{Mal}(X),\ \varphi^*\omega = \omega$.

Example $X = \frac{\partial}{\partial x_1} + x_3\frac{\partial}{\partial x_2} + E(x_1, x_2, x_3)\frac{\partial}{\partial x_3}$

θ is a 2-form such that $d\theta = 0$, $i_X\theta = 0$

$$\mathrm{Mal}(X) \subset \{\varphi \mid \varphi^*X = X, \varphi^*dx_1 = dx_1, \varphi^*\theta = \theta\}$$

$$\mathrm{Gal}(X/\mathbb{C}(x_1)) = \mathrm{Mal}(X) \cap \{\varphi \mid \varphi^*x_1 = x_1\}$$

Example $X = \frac{\partial}{\partial x_1} + x_3 \frac{\partial}{\partial x_2} + E(x)\frac{\partial}{\partial x_3}$

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In local coordinates x_1, y, z such that $X = \frac{\partial}{\partial x_1}$ and $\theta = dy \wedge dz$,

$\varphi \in \mathrm{Mal}(X) \Longrightarrow \varphi(x_1, y, z) = (x_1 + c, f(y,z), g(y,z))$ with $\dfrac{\partial(f,g)}{\partial(y,z)} = 1$

$\varphi \in \mathrm{Gal}(X/\mathbb{C}(x_1)) \Longrightarrow \ldots\ldots$ with $\quad c = 0$

Examples : Painlevé equations, $P_N(\alpha)$

- a vector field $X_N(\alpha) = \frac{\partial}{\partial x_1} + x_3 \frac{\partial}{\partial x_2} + (\ldots\ldots) \frac{\partial}{\partial x_3}$
- a 2-form $\theta_{N,\alpha}$ s.t. $d\theta_{N,\alpha} = 0$ and $i_{X_N(\alpha)}\theta_{N,\alpha} = 0$

Theorem

$$\mathrm{Mal}(X_N(\alpha)) = \{\varphi \mid \varphi^* X_N(\alpha) = X_N(\alpha), \varphi^* dx_1 = dx_1, \varphi^*\theta = \theta\}$$

- $N = 1$
- $N = 6$ *except for Picard parameters (Cantat-Loray)*
- $N = 2$, $\alpha \in \frac{1}{2}\mathbb{Z}$ *(+ Weil)*
- *Any* N, α *general (+ Davy)*

These are examples of “big enough” Galois groupoids.

- (M_i, X_i), $i = 1 \ldots n$, and $(M^{[n]}, X^{[n]})$ as before.
- $\pi_i : M_i \to \mathbb{A}^1$ s.t. $d\pi(X_i) = \frac{\partial}{\partial x_1}$
- θ_i a closed m_i-form on M_i with $i_{X_i}\theta_i = 0$.

Theorem

Assume $\mathrm{Mal}(X_i) = \{\varphi \mid \varphi^* X_i = X_i, \varphi^* dx_1 = dx_1, \varphi^* \theta_i = \theta_i\}$

If $V \subsetneq M^{[n]}$ is a $X^{[n]}$-invariant subvariety then

- $\exists\ i$, $pr_i(V) \subset M_i$ *has dimension* $< m$
- $\exists\ i < j$, $pr_{i,j}(V) \subset M_i \underset{\mathbb{A}^1}{\times} M_j$ *has dimension* m.

The proof by induction on n

Assume that projections $pr_1 : V \to M$ et $pr_{2,\dots n} : V \to M^{[n-1]}$ are dominant.

Theorem

If $\rho : (M, X) \dashrightarrow (N, Y)$ is rational, dominant and $d\rho(X) = Y$ then ρ induced a dominant morphism $\rho_* : \mathrm{Mal}(X) \to \mathrm{Mal}(Y)$.

Assume all pr_i are dominant.
One gets dominant projections $(pr_i)_* : \mathrm{Gal}(X^{[n]}/\mathbb{A}^1) \to \mathrm{Gal}(X_i/\mathbb{A}^1)$
and a inclusion $\mathrm{Gal}(X^{[n]}/\mathbb{A}^1) \subset \mathrm{Gal}(X_1/\mathbb{A}^1) \times \ldots \times \mathrm{Gal}(X_n/\mathbb{A}^1)$

Lemma

$\mathrm{Gal}(X^{[n]}/\mathbb{A}^1) = \mathrm{Gal}(X_1/\mathbb{A}^1) \times \ldots \times \mathrm{Gal}(X_n/\mathbb{A}^1)$

• As $\mathrm{Gal}(X_i/\mathbb{A}^1)$ are simple and projections are onto, it is enough to prove it for $n = 2$.

• Lie, Cartan : if G_1 and G_2 are infinite dimensional Lie pseudogroup then there is no $H \subset G_1 \times G_2$ whose projections are finite and onto.

Lemma

$pr_{2,\ldots,n} : V \to M^{[n-1]}$ is generically finite.

Fibers give a finite dimensional family of subvarieties of dimension $< m-1$ in fibres of $M_i \to \mathbb{A}$ and invariant under $\mathrm{Gal}(X_i/\mathbb{A})$ ...
This pseudogroup acts transitively on germs of curves.

Lemma

If $\mathcal{F}$ is a codimension m $X^{[n]}$-invariant foliation on V then $\exists\ i > 1$ s.t. leaves are fibers of pr_i .

Same argument is used on $M^{[n-1]}$ to describe invariants foliations under the action of $\mathrm{Gal}(X^{[n-1]}/\mathbb{A}^1)$.

Apply this to the foliation of V by fibers of pr_1, the lemma proves the theorem.

Remark 1 The condition depends only on the underlying foliation, you can change the independent variable :
From P_1 you get Mal of

$$\left(x'' + (x')^3\right)^2 + 24zx(x')^6 - (x')^5 = 0 \quad ; \quad ' = d/dz$$

Adding $x' \neq 0$ should define a strongly minimal set.

Remark 2 If you know an algebraic solution of X, you can verify the hypothesis on $\mathrm{Mal}(X)$ using your computer ... or by hands

If $n \geq 2$, $P \in \mathbb{C}(x, y)$ with $P(x, 0)$ having a pole of order k with $1 \leq k \leq n+3$, then Mal of

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Thank you for your attention